

# Gravity and compactified branes in matrix models

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## Abstract

A mechanism for emergent gravity on brane solutions in Yang-Mills matrix models is exhibited. Newtonian gravity and a partial relation between the Einstein tensor and the energy-momentum tensor can arise from the basic matrix model action, without invoking an Einstein-Hilbert-type term. The key requirements are compactified extra dimensions with extrinsic curvature  $\mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D$  and split noncommutativity, with a Poisson tensor  $\theta^{ab}$  linking the compact with the noncompact directions. The moduli of the compactification provide the dominant degrees of freedom for gravity, which are transmitted to the 4 noncompact directions via the Poisson tensor. The effective Newton constant is determined by the scale of noncommutativity and the compactification. This gravity theory is well suited for quantization, and argued to be perturbatively finite for the IKKT model. Since no compactification of the target space is needed, it might provide a way to avoid the landscape problem in string theory.

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## 1 Introduction

Matrix models such as the IKKT respectively IIB model [1] provide fascinating candidates for a quantum theory of fundamental interactions. Part of the appeal stems from the fact that geometry is not an input, but emerges on the solutions. For example, it is easy to see that flat noncommutative (NC) planes  $\mathbb{R}_\theta^{2n}$  arise as solutions. Similarly, branes with non-trivial geometry arise as NC sub-manifolds  $\mathcal{M} \subset \mathbb{R}^{10}$ , which can be interpreted as physical space-time. Their effective geometry is easily understood in the semi-classical limit [2, 3], in terms of a dynamical effective metric  $G_{ab}$  which is strongly reminiscent of the open string metric in the presence of a  $B$ -field [4]. This metric governs the kinematics of all propagating fields on the brane, and therefore describes gravity on the brane. Moreover, a relation with IIB supergravity or superstring theory<sup>2</sup> has been conjectured and verified to a certain extent [1, 5–9]. On the

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<sup>2</sup>for related work on the BFSS model [10] see e.g. [11–14].

other hand, the IKKT model can equivalently be viewed as  $\mathcal{N} = 4$  supersymmetric Yang-Mills (SYM) theory on  $\mathbb{R}_\theta^4$ , and is thus (expected to be) perturbatively finite in 4 dimensions. Combining these two points of view strongly suggests that the model should provide a quantum theory of fundamental interactions including gravity in 4 dimensions<sup>3</sup>.

However, a single 4-dimensional brane  $\mathcal{M}^4 \subset \mathbb{R}^{10}$  is clearly too simple to reproduce the rich spectrum of phenomena in nature. In order to recover e.g. the standard model, additional structure is needed. One possible origin of such additional structure are compactified extra dimensions, as considered in string theory. By considering intersecting branes and compactified extra dimensions in the matrix model, it is indeed possible to obtain chiral fermions and recover the basic structure of the standard model [15], adapting ideas from string theory [16].

The main point of the present paper is to show that the geometrical degrees of freedom provided by compactified extra dimensions also play a key role for the effective (emergent) gravity on such branes. Note that although the geometry of the branes is very clear, the dynamics of this geometry has not been well understood, and it is not evident that general relativity (GR) is recovered on the brane. A priori there is no Einstein-Hilbert term in the matrix model, which should be induced however by quantum effects. Thus an induced gravity mechanism is conceivable, but this is delicate. Moreover, the geometrical equations of motion are higher-order in the presence of an induced Einstein-Hilbert term, and it is not evident what type of gravitational solutions – Einstein or harmonic [3] – are appropriate<sup>4</sup>.

On the other hand, the brane carries a Poisson structure  $\theta^{ab}$  resp. a  $B$ -field which is an integral part of the effective metric, and which moreover constitutes a background where (local) Lorentz invariance is spontaneously broken. This breaking of Lorentz invariance is almost invisible, since  $\theta^{ab}$  is completely absorbed in the effective metric  $G^{ab}$  and does not explicitly couple to any field at least at tree level [3]. Nevertheless, the basic degrees of freedom which encode the geometry are not the metric as in GR, but the embedding of the brane along with  $\theta^{ab}$ . Disentangling these degrees of freedom and understanding their significance is far from trivial.

In the present paper, we exhibit a novel mechanism for gravity on branes with compactified extra dimensions  $\mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^{10}$ , which is based on the bare matrix model action without requiring the presence of an induced Einstein-Hilbert type term. The mechanism is therefore very robust, and is expected to give the dominant contribution to 4-dimensional gravity on the brane. Although the brane gravity is *not* equivalent to GR, essential features of GR are recovered, in particular a (partial) relation between the Einstein tensor and the energy-momentum (e-m) tensor. The precise coupling depends on the compactification, and we focus mainly on the simplest case of  $\mathcal{M}^4 \times T^2$  in the present paper. Newtonian gravity is recovered, with an effective Newton constant that is determined by the scale of noncommutativity and the compactification. Although the post-Newtonian corrections are anisotropic in this simple compactification due to a violation of local Lorentz invariance, we argue that more sophisticated backgrounds should alleviate this problem. It then appears possible that a viable gravity could be obtained in this manner. As a bonus, we will argue that the physics of vacuum energy is different from GR, which could have important consequences for cosmology. We give an argument that the usual fine-tuning problem associated with quantum mechanical

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<sup>3</sup>The relation with string theory is not the main topic of this paper, and we focus on the 4-dimensional brane geometry. However we will argue that the bulk should be understood in a holographic manner.

<sup>4</sup>Another rather different approach to obtain gravity from the IKKT model has been proposed in [17]. Its significance and relation with the solutions considered here is not clear to the author.

vacuum fluctuations leading to the cosmological constant problem should not arise here.

The mechanism discussed in this paper depends on two crucial conditions:

1. non-vanishing extrinsic curvature of the embedding  $\mathcal{M} = \mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^{10}$  predominantly due to  $\mathcal{K} \subset \mathbb{R}^{10}$ , and
2. "split noncommutativity" [18] where the Poisson tensor  $\theta^{ac}$  links the noncompact with the compact spaces. This transmutes perturbations of the moduli of  $\mathcal{K}$  into perturbations of the effective metric.

Let us discuss this in more detail. In GR, gravity is characterized the intrinsic geometry of the 4-dimensional space-time manifold, while its specific realization – via an isometric embedding or as abstract manifold – is irrelevant. For branes arising as solutions of matrix models, this is not the case. First and foremost, the fundamental degrees of freedom are different, given not by the metric but by brane embeddings and their fluctuations, as well as a Poisson structure. The metric is a derived quantity. One key observation [19] is that linearized embedding fluctuations couple linearly to the energy-momentum tensor in the presence of *extrinsic curvature* of  $\mathcal{M} \subset \mathbb{R}^{10}$ , leading to (Newtonian, at least) gravity. However this required somewhat ad-hoc assumptions. The new observation in the present paper is that the mechanism naturally applies for compactified extra dimensions, leading to a (partial) relation between the Einstein tensor and the energy-momentum tensor at least in the linearized approximation. Moreover, fluctuations of the radial moduli of the compactified extra dimensions  $\mathcal{K}$  are transmuted via  $\theta^{ab}$  to 4-dimensional metric fluctuations. Here  $\mathcal{K}$  is in a sense rotating and stabilized by angular momentum, given e.g. by a torus  $\mathcal{K} = T^n$  with light-like compactification. Such solutions of the matrix model have been given recently [18].

It is important to emphasize that the effective gravity is indeed 4-dimensional, even though the brane  $\mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D$  is embedded in a higher-dimensional non-compact target space. This is in contrast to the conventional picture in string theory, where gravity is supposed to originate from closed strings which propagate in 10 dimensions, leading to a 10-dimensional Newton law. The crucial point is that brane gravity emerges here<sup>5</sup> entirely within the open string sector on the brane with a non-vanishing Poisson structure resp.  $B \neq 0$ , while the 10-dimensional bulk arises in a holographic manner (cf. [21]; this can be seen in the matrix model at the one-loop level [5, 9]). This is very welcome, since one can now discard the vast landscape of 10-dimensional compactifications, and study the mini-landscape of embedded compactified branes embedded in  $\mathbb{R}^{10}$ , as described by the IKKT model. This is a well-posed problem which should have a clear non-perturbative answer.

Finally, although the backgrounds  $\mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D$  under consideration here are at least 6-dimensional at low energies, they behave as 4-dimensional spaces in the UV due to noncommutativity [18]. This means that the finiteness of the IKKT model viewed as  $\mathcal{N} = 4$  SYM in 4 dimensions still applies in the UV limit of the backgrounds under consideration, so that the model is expected to be UV finite, and without pathological UV/IR mixing. This is in stark contrast with commutative SYM and supergravity, which would not be UV complete and require a UV completion via string theory. The matrix model approach therefore promises to

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<sup>5</sup>This was also realized recently in a different approach to emergent gravity [20], however their matrix model still requires UV completion via string theory. The present model is claimed to be complete. The origin for the 4-dimensional behavior is also very different from e.g. the DGP mechanism [22], which is based on a combination of brane and bulk physics without a  $B$  field.

provide a consistent and self-contained approach towards a quantum theory of all fundamental interactions, and might resolve the landscape problem in string theory. This certainly justifies further studies.

This paper is organized as follows. After a brief review of basic aspects of branes in matrix models, we discuss in detail perturbations of branes and the associated curvature perturbations. The dynamics of these metric fluctuations coupled to matter is derived in section 4. Next the compactification is discussed in some detail, and the role of moduli as mediators of gravity is exhibited. The effective 4-dimensional gravity is then derived, focusing on the case of toroidal compactifications. In particular the background  $M^4 \times T^2$  is worked out in detail. In section 5 we discuss possible generalizations of the compactification, arguing that more realistic backgrounds can be found. However, the study of such generalized compactifications is left for future work.

Throughout this paper, a slightly cumbersome but explicit index notation is used. This is done in order not to hide things under the carpet, and a more elegant formulation can be given eventually.

## 2 Matrix models and their geometry

We briefly collect the essential ingredients of the matrix model framework and its effective geometry, referring to the recent review [3] for more details.

### 2.1 The IKKT model and related matrix models

The starting point is given by a matrix model of Yang-Mills type,

$$S_{\text{YM}} = -\frac{\Lambda_0^4}{4} \text{Tr} \left( [X^A, X^B][X^C, X^D] \eta_{AC} \eta_{BD} + 2\bar{\Psi} \gamma_A [X^A, \Psi] \right) \quad (2.1)$$

where the  $X^A$  are Hermitian matrices, i.e. operators acting on a separable Hilbert space  $\mathcal{H}$ . The indices of the matrices run from 0 to  $D-1$ , and will be raised or lowered with the invariant tensor  $\eta_{AB}$  of  $SO(D-1, 1)$ . Although this paper is mostly concerned with the bosonic sector, we focus on the maximally supersymmetric IKKT or IIB model [1] with  $D = 10$ , which is best suited for quantization. Then  $\Psi$  is a matrix-valued Majorana Weyl spinor of  $SO(9, 1)$ . The model enjoys the fundamental gauge symmetry

$$X^A \rightarrow U^{-1} X^A U, \quad \Psi \rightarrow U^{-1} \Psi U, \quad U \in U(\mathcal{H}) \quad (2.2)$$

as well as the 10-dimensional Poincaré symmetry

$$\begin{aligned} X^A &\rightarrow \Lambda(g)^A_B X^B, & \Psi_\alpha &\rightarrow \tilde{\pi}(g)_\alpha^\beta \Psi_\beta, & g &\in \widetilde{SO}(9, 1), \\ X^A &\rightarrow X^A + c^A \mathbf{1}, & & & c^A &\in \mathbb{R}^{10} \end{aligned} \quad (2.3)$$

and a  $\mathcal{N} = 2$  matrix supersymmetry [1]. The tilde indicates the corresponding spin group. We also introduced a parameter  $\Lambda_0$  of dimension  $[L]^{-1}$ , so that the  $X^A$  have dimension length, corresponding to the (trivial) scaling symmetry

$$X^A \rightarrow \alpha X^A, \quad \Psi \rightarrow \alpha^{3/2} \Psi, \quad \Lambda_0 \rightarrow \alpha^{-1} \Lambda_0. \quad (2.4)$$

On backgrounds with  $S = 0$  there is also a non-trivial scaling symmetry  $X^A \rightarrow \alpha X^A$ . We define the matrix Laplacian as

$$\square \Phi := [X_B, [X^B, \Phi]] \quad (2.5)$$

for any matrix  $\Phi \in \mathcal{L}(\mathcal{H})$ . Then the equations of motion of the model take the following form

$$\square X^A = [X_B, [X^B, X^A]] = 0 \quad (2.6)$$

for all  $A$ , assuming  $\Psi = 0$ .

## 2.2 Noncommutative branes and their geometry

Now we focus on matrix configurations which describe embedded noncommutative (NC) branes. This means that the  $X^A$  can be interpreted as quantized embedding functions [3]

$$X^A \sim x^A : \quad \mathcal{M}^{2n} \hookrightarrow \mathbb{R}^{10} \quad (2.7)$$

of a  $2n$ -dimensional submanifold of  $\mathbb{R}^{10}$ . More precisely, there should be some quantization map  $\mathcal{I} : \mathcal{C}(\mathcal{M}) \rightarrow \mathcal{A} \subset L(\mathcal{H})$  which maps classical functions on  $\mathcal{M}$  to a noncommutative (matrix) algebra of functions, such that commutators can be interpreted as quantized Poisson brackets. In the semi-classical limit indicated by  $\sim$ , matrices are identified with functions via  $\mathcal{I}$ , and commutators are replaced by Poisson brackets; for a more extensive introduction see e.g. [3, 23]. One can then locally choose  $2n$  independent coordinate functions  $x^a$ ,  $a = 1, \dots, 2n$  among the  $x^A$ , and their commutators

$$[X^a, X^b] \sim i\{x^a, x^b\} = i\theta^{ab}(x) \quad (2.8)$$

encode a quantized Poisson structure on  $(\mathcal{M}^{2n}, \theta^{ab})$ . These  $\theta^{ab}$  have dimension  $[L^2]$  and set a typical scale of noncommutativity  $\Lambda_{\text{NC}}^{-2}$ . We will assume that  $\theta^{ab}$  is non-degenerate<sup>6</sup>, so that the inverse matrix  $\theta_{ab}^{-1}$  defines a symplectic form on  $\mathcal{M}^{2n} \subset \mathbb{R}^{10}$ . This submanifold is equipped with the induced metric

$$g_{ab}(x) = \partial_a x^A \partial_b x_A \quad (2.9)$$

which is the pull-back of  $\eta_{AB}$ . However, this is *not* the effective metric on  $\mathcal{M}$ . To understand the effective metric and gravity, we need to consider matter on the brane  $\mathcal{M}$ . Bosonic matter or fields arise from nonabelian fluctuations of the matrices around a stack  $X^A \otimes \mathbb{1}_n$  of coinciding branes, while fermionic matter arises from  $\Psi$  in (2.1). It turns out that in the semi-classical limit, the effective action for such fields is governed by a universal effective metric  $G^{ab}$ . It can be obtained most easily by considering the action of an additional scalar field  $\phi$  coupled to the matrix model in a gauge-invariant way, with action

$$\begin{aligned} S[\phi] &= -\frac{\Lambda_0^4}{2} \text{Tr} [X_A, \phi] [X^A, \phi] \sim \frac{\Lambda_0^4}{2(2\pi)^n} \int d^{2n}x \sqrt{|\theta^{-1}|} \theta^{aa'} \theta^{bb'} g_{a'b'} \partial_a \phi \partial_b \phi \\ &= \frac{\Lambda_0^4}{2(2\pi)^n} \int d^{2n}x \sqrt{|G|} G^{ab} \partial_a \phi \partial_b \phi. \end{aligned} \quad (2.10)$$

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<sup>6</sup>If the Poisson structure is degenerate, then fluctuations propagate only along the symplectic leaves.

Therefore the effective metric is given by [2]

$$\begin{aligned} G^{ab} &= e^{-\sigma} \theta^{aa'} \theta^{bb'} g_{a'b'} , \\ e^{-\sigma} &= \left( \frac{\det \theta_{ab}^{-1}}{\det G_{ab}} \right)^{\frac{1}{2}} = \left( \frac{\det \theta_{ab}^{-1}}{\det g_{ab}} \right)^{\frac{1}{2(n-1)}} \end{aligned} \quad (2.11)$$

which is very much like the open string metric on D-branes with a  $B$ -field [4]. Let us briefly discuss the scales and dimensions. Clearly  $e^{-\sigma}$  characterizes the NC scale<sup>7</sup>, and  $\Lambda_0$  is related to  $e^{-\sigma}$  via the Yang-Mills coupling constant [3]

$$\Lambda_0^4 e^\sigma = \frac{1}{g_{\text{YM}}^2} \quad (2.12)$$

which governs the  $SU(n)$  sector. The transversal matrices resp. their fluctuations  $\phi^i \equiv \delta X^i$  will be considered as perturbation of the embedding, with dimension  $\dim \phi^i = [L]$ . On the other hand, nonabelian fluctuations of the transversal matrices should be viewed as scalar fields, via the identification

$$\varphi = \Lambda_0^2 \phi, \quad S[\phi] \sim \frac{1}{2(2\pi)^n} \int d^{2n}x \sqrt{|G|} G^{ab} \partial_a \varphi \partial_b \varphi. \quad (2.13)$$

Then  $\dim \varphi^i = [L^{-1}]$ , and the energy-momentum (e-m) tensor is recovered correctly.

The important point is that the metric  $G$  governs the semi-classical limit of all fields propagating on  $\mathcal{M}$  including scalar fields, non-Abelian gauge fields and fermions [2, 24]. This means that  $G$  must be interpreted as gravitational metric. Therefore the model provides a dynamical gravity theory, realized on dynamically determined branes  $\mathcal{M} \subset \mathbb{R}^{10}$  governed by the action (2.1). To understand the dynamics of the geometry in more detail, the following result is useful [3]: the matrix Laplace operator reduces in the semi-classical limit to the covariant Laplace operator

$$\square \Phi = [X_A, [X^A, \Phi]] \sim -e^\sigma \square_G \phi \quad (2.14)$$

acting on scalar fields  $\Phi \sim \phi$ . In particular, the matrix equations of motion (2.6) take the simple form

$$0 = \square X^A \sim -e^\sigma \square_G x^A. \quad (2.15)$$

This means that the embedding functions  $x^A \sim X^A$  are harmonic functions with respect to  $G$ . Furthermore, the bosonic matrix model action (2.1) can be written in the semi-classical limit as follows

$$S_{\text{YM}} \sim \frac{\Lambda_0^4}{4(2\pi)^{2n}} \int d^{2n}x \sqrt{|\theta^{-1}|} \gamma^{ab} g_{ab}. \quad (2.16)$$

Here we introduce the conformally equivalent metric<sup>8</sup>

$$\gamma^{ab} = \theta^{aa'} \theta^{bb'} g_{a'b'} = e^\sigma G^{ab} \quad (2.17)$$

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<sup>7</sup>Thus  $e^{-\sigma}$  is dimensionful, and could be made dimensionless by absorbing suitable powers of  $\Lambda_0$ . However we stick to the above conventions to keep the formulae simple. Note also that the scaling dimensions of  $G$  and  $g$  are distinct and rather peculiar for  $\dim \mathcal{M} \neq 4$ . This reflects the fact that the trace is related to the symplectic volume form.

<sup>8</sup>More abstractly, this can be stated as  $(\alpha, \beta)_\gamma = (i_\alpha \theta, i_\beta \theta)_g$  where  $\theta = \frac{1}{2} \theta^{ab} \partial_a \wedge \partial_b$ .

which satisfies

$$\sqrt{|\theta^{-1}|}\gamma^{ab} = \sqrt{|G|}G^{ab}. \quad (2.18)$$

Note also that  $\theta^{ab}\partial_b$  is somewhat reminiscent of a frame, which could be made more suggestive in the form

$$e^{(a)} = \{x^a, \cdot\} = \theta^{ac}\partial_c, \quad \gamma^{ab} = (e^{(a)}, e^{(b)})_g \quad (2.19)$$

cf. [25]. However the analogy is somewhat misleading for proper submanifolds, because  $g_{ab}$  is dynamical and not flat in general.

### 2.2.1 Perturbations of the matrix geometry

Now consider a brane  $\mathcal{M}^{2n} \subset \mathbb{R}^D$  obtained as a perturbation

$$X^A = \bar{X}^A + \delta X^A \quad (2.20)$$

of some background brane  $\bar{\mathcal{M}}$ , defined in terms of matrices  $\bar{X}^A$  as above. We want to understand the metrics  $G^{ab}$  and  $g_{ab}$  on  $\mathcal{M}$  as deformations of  $\bar{G}^{ab}$  and  $\bar{g}_{ab}$  on  $\bar{\mathcal{M}}$ . In the semi-classical limit, the perturbation can be split into  $D - 2n$  transversal perturbations  $\delta_\perp X^A \sim \phi^A$  and  $2n$  tangential perturbation  $\delta_\parallel X^A \sim \mathcal{A}^A$ , defined by

$$\phi_A \partial_a \bar{x}^A = 0, \quad (2.21)$$

$$\mathcal{A}^A = \mathcal{A}^a \partial_a \bar{x}^A. \quad (2.22)$$

To make this more transparent, consider some point  $p \in \bar{\mathcal{M}}$ . We can assume using Poincaré invariance that  $p$  is at the origin, and the tangent plane  $T_p \bar{\mathcal{M}}$  is embedded along the first  $2n$  Cartesian embedding coordinates

$$\bar{x}^A = (\bar{x}^a, \bar{y}^i) \quad \text{with} \quad y^i|_p = 0, \quad \partial_a|_p \bar{y}^i = 0. \quad (2.23)$$

This defines “normal embedding coordinates” (NEC)  $\bar{x}^a \sim \bar{X}^a$ , which can be used both on  $\bar{\mathcal{M}}$  and  $p \in \mathcal{M}$  near  $p$ ; hence we omit the bar from now on. They are normal coordinates corresponding to the connection  $\nabla^{(g)}$  defined by the embedding metric  $g$ . However, we will mostly use the connection  $\nabla \equiv \nabla^{(G)}$  defined by the effective metric  $\bar{G}$  for the following. Then the transversal variations are given by the  $\phi^A = (0, \phi^i)$ , and the tangential variations by  $\mathcal{A}^A = (\mathcal{A}^a, 0)$ . If the matrix model is viewed as NC gauge theory, then these variations  $\mathcal{A}^a$  and  $\phi^i$  can be interpreted in terms of “would-be”  $U(1)$  gauge fields and scalar fields on  $\bar{\mathcal{M}}$ ; this is useful for perturbative computations. The tangential variations lead to a perturbed Poisson structure on  $\mathcal{M}$

$$\theta^{ab}(x) = \bar{\theta}^{ab} + \delta\theta^{ab}(x), \quad \delta\theta^{ab}(x) = \bar{\theta}^{ac}\bar{\theta}^{bd}\delta\theta_{cd}^{-1}, \quad (2.24)$$

which can be parametrized in terms of the a “would-be”  $U(1)$  gauge field

$$\delta\theta_{ab}^{-1} = F_{ab} = \nabla_a \delta A_b - \nabla_b \delta A_a. \quad (2.25)$$



The transversal embedding perturbation  $\delta_\perp X^A \sim \phi^A$  satisfy the constraint (2.21), which implies

$$\partial_b \phi_A \partial_a \bar{x}^A = -\phi_A \nabla_b \partial_a \bar{x}^A = -\phi_A K_{ab}^A. \quad (2.26)$$

Here

$$K_{ab}^A = \nabla_a \partial_b \bar{x}^A = K_{ba}^A \quad (2.27)$$

is the the 2nd fundamental form, which characterizes the exterior curvature of  $\mathcal{M} \subset \mathbb{R}^D$ , and will play a central role in the following. Notice that  $K_{ab}^A$  takes values in the normal bundle  $N\bar{\mathcal{M}}$  provided  $\nabla_c \bar{\theta}^{ab} = 0$  (since then the connections defined by the induced and the effective metrics on  $\bar{\mathcal{M}}$  coincide, so that  $\partial_a x_A \nabla_b \partial_c x^A = 0$ ). The metric fluctuations are obtained as

$$\begin{aligned} g_{ab} &= \bar{g}_{ab} + \delta g_{ab}, \\ \delta g_{ab} &= \partial_a \phi_A \partial_b \bar{x}^A + \partial_a \bar{x}_A \partial_b \phi^A = -2\phi_A K_{ab}^A, \end{aligned} \quad (2.28)$$

$$\begin{aligned} \gamma^{ab} &= \bar{\gamma}^{ab} + \delta \gamma^{ab}, \\ \delta \gamma^{ab} &= \bar{\theta}^{ac} \bar{\theta}^{bd} \delta g_{cd} - \bar{\theta}^{ac} F_{cc'} \gamma^{bc'} - \bar{\theta}^{bc} F_{cc'} \gamma^{ac'}, \end{aligned} \quad (2.29)$$

$$\delta \sigma = \frac{1}{2} (G^{ab} \delta G_{ab} + \bar{\theta}^{ab} F_{ab}) = \frac{1}{2(n-1)} (g^{ab} \delta g_{ab} + \bar{\theta}^{ab} F_{ab}), \quad (2.30)$$

using (2.11) in the last equation. There are also quadratic terms in  $\phi, F$  which are omitted. Hence the perturbation of the effective metric  $G^{ab}$  is given by

$$\begin{aligned} \delta G^{ab} &= e^{-\sigma} \delta \gamma^{ab} - G^{ab} \delta \sigma \\ &= -2e^{-\sigma} \bar{\theta}^{aa'} \bar{\theta}^{bb'} \Pi_{ab}^{cd} K_{cd}^A \phi_A - \bar{\theta}^{ac} F_{cd} G^{bd} - \bar{\theta}^{bc} F_{cd} G^{ad} - \frac{G^{ab}}{2(n-1)} (\bar{\theta} F) \end{aligned} \quad (2.31)$$

using the abbreviations

$$\begin{aligned} (\theta F) &= \theta^{ab} F_{ab} \\ \Pi_{ab}^{cd} &= \delta_{ab}^{cd} - \frac{g_{ab} g^{cd}}{2(n-1)}. \end{aligned} \quad (2.32)$$

## 2.3 Curvature perturbations

Now we take advantage of the special coordinates  $x^a \sim X^a$ ,  $a = 1, \dots, 2n$  provided by some suitable subset of the matrices  $X^A$ , for example the NEC defined above. The equations of motion  $\square X^a = 0$  in vacuum implies that these coordinates satisfy the harmonic gauge condition,

$$0 \sim \square_G x^a = -\Gamma^a = |G_{cd}|^{-1/2} \partial_b (\sqrt{|G_{cd}|} G^{ba}). \quad (2.33)$$

For the metric fluctuations  $h_{ab} = \delta G_{ab}$  this implies the harmonic gauge condition

$$\partial^b h_{ab} - \frac{1}{2} \partial_a h = 0. \quad (2.34)$$

Then the perturbation of the Ricci tensor around a general background [26]

$$\delta R_{ab} = -\frac{1}{2} \bar{\nabla}_a \partial_b h - \frac{1}{2} \bar{\square}_G h_{ab} + \bar{\nabla}_{(a} \bar{\nabla}^d h_{b)d} + \bar{R}_{ab}^c{}^d h_{cd} + \bar{R}_a^c h_{bc} + \bar{R}_b^c h_{ac} \quad (2.35)$$

simplifies as

$$\delta_h R_{ab} = -\frac{1}{2}\bar{\square}_G h_{ab} + \mathcal{O}(\bar{R}). \quad (2.36)$$

Similarly, the perturbations of the Einstein tensor  $\mathcal{G}_{ab} = R_{ab} - \frac{1}{2}G_{ab}R$  can be written as follows

$$\delta\mathcal{G}_{ab} = -\frac{1}{2}\bar{\square}_G \left( \delta G_{ab} - \frac{1}{2}\bar{G}_{ab}(G^{cd}\delta G_{cd}) \right) + \mathcal{O}(\bar{R}). \quad (2.37)$$

Noting that  $\delta G_{ab} = -\bar{G}_{aa'}\bar{G}_{bb'}\delta G^{a'b'}$  while  $\delta\mathcal{G}_{ab}$  is a tensor, this can be written as

$$\begin{aligned} \delta\mathcal{G}^{ab} &= \frac{1}{2} \left( \bar{\square}_G \delta G^{ab} + \frac{1}{2}\bar{G}^{ab}\bar{\square}_G(\bar{G}^{cd}\delta G_{cd}) \right) + \mathcal{O}(\bar{R}) \\ &= \frac{1}{2} \left( \bar{\square}_G(e^{-\sigma}\delta\gamma^{ab} - \bar{G}^{ab}\delta\sigma) + \bar{G}^{ab}\bar{\square}_G(\delta\sigma - \frac{1}{2}(\bar{\theta}F)) \right) + \mathcal{O}(\bar{R}) \\ &= \frac{1}{2} \left( e^{-\sigma}\bar{\square}_G\delta\gamma^{ab} - \frac{1}{2}\bar{G}^{ab}\bar{\square}_G(\theta F) \right) + \mathcal{O}(\bar{R}) \end{aligned} \quad (2.38)$$

using (2.30), assuming  $\sigma = \text{const}$  for the background. We will mostly use this equation for a locally adapted flat background, corresponding to normal coordinates. Then the curvature corrections drop out, and this relation allows to compute the full Einstein tensor.

### 3 Gravity on higher-dimensional branes

To understand how matter affects the geometry, we now study the dynamics of these geometrical modes. The goal is to show that the vacuum geometry is Ricci-flat to a good approximation, and matter couples to the Einstein tensor in a way similar to general relativity.

Assume  $\mathcal{M} \subset \mathbb{R}^D$  is a brane as above, described by  $X^A \sim x^A$ . Unlike in general relativity, the dynamics is governed by the effective action (2.16), where the geometric perturbations can be organized into transversal and parallel fluctuations  $\delta_\perp x^A = \phi^A$  resp.  $\delta\theta_{ab}^{-1} = F_{ab}$  as above. We expand the action (2.16) in  $\phi^A$  and  $F_{ab}$ . To first order, one obtains

$$S^{(1)}[\phi, F] = \frac{\Lambda_0^4}{4(2\pi)^n} \int d^{2n}x \sqrt{|G|} \left( 2F_{ab}(e^\sigma G^{bc}\theta_{cd}^{-1}G^{da} - \frac{1}{4}\theta^{ab}(G \cdot g)) - 4\phi_A K_{ab}^A G^{ab} + \lambda^a \phi_A \partial_a \bar{x}^A \right) \quad (3.1)$$

where  $(G \cdot g) \equiv G^{cd}g_{cd}$ . Now we take matter into account. Physical fields and matter arises in the matrix model from nonabelian fluctuations of the bosonic matrices  $X^A$  around the background, and from the fermionic matrices  $\Psi$ . Since they couple in the standard way to the effective metric  $G$ , the variation of their action with respect to fluctuations of the geometry can be written as<sup>9</sup>

$$\delta S_{\text{matter}} = \frac{1}{2(2\pi)^n} \int d^{2n}x \sqrt{G} T_{ab} \delta G^{ab} \quad (3.2)$$

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<sup>9</sup>Notice that there is no scale factor  $\Lambda_0$ ; it is absorbed in  $T_{ab}$  upon recasting the nonabelian fields into physical form, introducing dimensions as in (2.11). The normalization is chosen to avoid factors  $2\pi$  later.

where  $\delta G^{ab}$  is given by (2.31). To obtain the equations of motion for the geometry, we need the variation of the matter action

$$\begin{aligned}\int d^{2n}x \sqrt{G} T_{ab} \delta_\phi G^{ab} &= - \int d^{2n}x \sqrt{G} \phi_A (e^{-\sigma} T_{ab} \theta^{aa'} \theta^{bb'} \Pi_{a'b'}^{cd} K_{cd}^A) \\ \int d^{2n}x \sqrt{G} T_{ab} \delta_A G^{ab} &= \int d^{2n}x \sqrt{G} \delta A_d (2T_{ab} \nabla^b \theta^{da} + e^{-\sigma} \theta^{ac} \nabla_c (e^\sigma T_{ab}) G^{db} + \frac{e^{-\sigma}}{n-1} \theta^{cd} \nabla_c T)\end{aligned}$$

where  $T = \gamma^{ab} T_{ab}$ , dropping a term proportional to<sup>10</sup>  $\nabla^a T_{ab}$ , and using the identity [3]

$$\nabla_a (e^{-\sigma} \theta^{ab}) = 0. \quad (3.3)$$

Instead of writing an equation for  $\theta^{ab}$  in the form  $\nabla^d (e^\sigma \theta_{cd}^{-1}) - \frac{1}{4} G_{bc} \theta^{ab} \partial_a (G \cdot g) = \mathcal{O}(T)$  as in [2], it is more useful to rewrite  $S^{(1)}[\phi, F]$  using the identity (B.9) as

$$\begin{aligned}S^{(1)}[F] &= \frac{\Lambda_0^4}{(2\pi)^n} \int d^{2n}x \left( A_a \nabla_b (\sqrt{|G|} G^{bc} T_{cd}^{\text{geom}} \theta^{da}) \right) \\ &= -\frac{\Lambda_0^4}{(2\pi)^n} \int d^{2n}x \sqrt{G} A_a \left( e^\sigma G^{ab} G^{de} \nabla_e^{(g)} \theta_{db}^{-1} \right)\end{aligned} \quad (3.4)$$

where  $T^{\text{geom}}$  is defined in (B.5). We then obtain the equations of motion

$$G^{ab} \nabla_a^{(g)} \nabla_b^{(g)} x^A = -\Lambda_0^{-4} e^{-\sigma} K_{cd}^A \Pi_{a'b'}^{cd} \theta^{aa'} \theta^{bb'} T_{ab}, \quad (3.5)$$

$$G^{ce} \nabla_e^{(g)} \theta_{cb}^{-1} = \Lambda_0^{-4} e^{-\sigma} \left( 2T_{ae} G^{ec} \nabla_c \theta^{da} G_{db} + e^{-\sigma} \theta^{ac} \nabla_c (e^\sigma T_{ab}) + \frac{e^{-\sigma}}{(n-1)} G_{bd} \theta^{cd} \nabla_c T \right), \quad (3.6)$$

which are valid for arbitrary geometries. Note that (3.5) relates the mean extrinsic curvature  $K^A = G^{ab} K_{ab}^A$  to the energy-momentum tensor. As usual, the Maxwell-like equation (3.6) implies via the Bianchi identity a wave equation

$$G^{ab} \nabla_a^{(g)} \nabla_b^{(g)} \theta_{cd}^{-1} = \nabla_c J_d - \nabla_c^{(g)} G^{ab} \nabla_b \theta_{ad}^{-1} - (c \leftrightarrow d) + \mathcal{O}(R[g]) \quad (3.7)$$

where  $J_a$  is defined by the rhs of (3.6).

### 3.1 Perturbed flat branes and linearized gravity

Now recall the two possible interpretations of the matrix model: A) as model for branes  $\mathcal{M}$  whose geometry is determined by  $\phi^i$  resp.  $F_{\mu\nu}$ , as discussed above and B) as a NC field theory on some given background, where  $\phi^i$  resp.  $F_{\mu\nu}$  are interpreted as  $U(1)$ -valued fields on a given background  $\bar{\mathcal{M}}$ . Let us now use B), which is more useful for the perturbation theory, and sufficient as long as the perturbations are small. As explained below, this can always be done by choosing a locally adapted background.

Thus consider some intrinsically flat background solution  $\bar{x}^A : \bar{\mathcal{M}} \hookrightarrow \mathbb{R}^{10}$  of the bare matrix model without matter, which satisfies  $\bar{\square} x^A = 0$ . We can furthermore assume  $\bar{\nabla}^{(g)} \bar{\theta}^{ab} = 0$  for the background solution without matter, since the intrinsic geometry is assumed to be flat;

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<sup>10</sup>The usual form of the conservation law is assumed, although it might be slightly modified in the NC case.

this implies  $\bar{\nabla}^{(g)}\bar{G} = 0 = \partial\bar{\sigma}$ , hence  $\bar{\nabla}^{(g)} = \bar{\nabla}^{(G)} \equiv \bar{\nabla}$ . We have in mind  $\bar{\mathcal{M}} = M^4 \times T^2$ , as given explicitly in sector 4.2. Now add perturbations  $\theta^{-1} = \bar{\theta}^{-1} + F$ ,  $x^A = \bar{x}^A + \phi^A$ . To derive the equations of motion, we need the second order variation of the action (2.16) on this background  $\bar{\mathcal{M}}$ , given by

$$S^{(2)}[\phi, F] = \frac{\Lambda_0^4}{4(2\pi)^n} \int d^{2n}x \sqrt{|G|} \left( e^\sigma \bar{G}^{aa'} \bar{G}^{bb'} F_{ab} F_{a'b'} + 2\bar{G}^{ab} \partial_a \phi_A \partial_b \phi^A + 2\phi^A m_{AB}^2 \phi^B \right. \\ \left. + 4\bar{\theta}^{ac} F_{cd} \bar{G}^{db} \phi_A K_{ab}^A \right) + S_{CS} \quad (3.8)$$

where

$$m_{AB}^2 = 2K_{ab}^A \theta^{aa'} \theta^{bb'} K_{a'b'}^B \quad (3.9)$$

using (2.29), and  $\lambda^a$  are Lagrangian multipliers which implement the constraint (2.21). The “would-be topological” term<sup>11</sup> [24]  $S_{CS} \sim \int \rho \langle F \wedge F, \theta \wedge \hat{\theta} \rangle - \frac{1}{2}(\hat{\theta} \rightarrow \eta\theta)$  is only relevant for the propagating gravitational modes considered in section 3.3, and can be dropped for  $\bar{\nabla}\bar{\theta}^{ab} = 0$ . The term

$$S_{\text{mix}} = \frac{1}{(2\pi)^n} \int d^{2n}x \sqrt{G} \bar{\theta}^{ac} F_{cd} \bar{G}^{db} \phi_A K_{ab}^A \\ = \frac{1}{(2\pi)^n} \int d^{2n}x \sqrt{G} T_{ab}[\phi] \delta_A G^{ab} = \frac{1}{(2\pi)^n} \int d^{2n}x \sqrt{G} T_{ab}[F] \delta_\phi G^{ab} \quad (3.10)$$

mixes the tangential and transversal perturbations, and can be written as a coupling to some effective induced energy-momentum tensors

$$T_{ab}[\phi] = -\frac{1}{2}\Lambda_0^4 \delta_\phi g_{ab} = \Lambda_0^4 \phi_A K_{ab}^A, \quad T_{ab}[\phi] \bar{G}^{ab} = 0 \\ T_{ab}[F] = \frac{1}{4}\Lambda_0^4 F_{ad} \bar{\theta}^{db} g_{db} + (a \leftrightarrow b) + \frac{1}{4}\Lambda_0^4 G_{ab} (F_{cd} \theta^{de} g_{ef} G^{fc}) \quad (3.11)$$

using the background equations of motion. We can then write  $S^{(2)}[\phi, F]$  as

$$S^{(2)}[\phi, F] = -\frac{\Lambda_0^4}{(2\pi)^n} \int d^{2n}x \sqrt{|G|} \left( A_d \bar{G}^{da} \bar{G}^{bc} \bar{\nabla}_c (e^\sigma F_{ab}) + \phi^A (\eta_{AB} \bar{\square} - m_{AB}^2) \phi^B \right) + S_{\text{mix}}$$

using (2.31). Note that  $\eta_{AB}$  will be positive for the transversal fluctuations  $\phi^A$ , so there is no stability problem. This gives the equations of motion

$$\bar{\square} \phi^A + m_{CB}^2 \eta^{AC} \phi^B = -\Lambda_0^{-4} e^{-\sigma} K_{cd}^A \Pi_{a'b'}^{cd} \bar{\theta}^{a'a} \bar{\theta}^{b'b} T_{ab}^{M+F} \quad (3.12) \\ \bar{\nabla}^b F_{ab} = 2\Lambda_0^{-4} e^{-\sigma} \left( \theta^{bc} \bar{\nabla}_c T_{ab}^{M+\phi} + G_{ad} \theta^{dc} \bar{\nabla}^b T_{cb}^{M+\phi} - \frac{1}{2n-2} G_{ad} \theta^{dc} \bar{\nabla}_c T^{M+\phi} \right)$$

where

$$T_{ab}^{M+\phi} = T_{ab} + T_{ab}[\phi], \quad \tilde{T}_{ab}^{M+F} = T_{ab} + T_{ab}[F]. \quad (3.13)$$

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<sup>11</sup>A simplified derivation of  $S_{CS}$  could be given along the lines of section 4 in [27].

This implies again a wave-equation via (3.7)

$$\bar{\square} F = \nabla \nabla (T + T[\phi]). \quad (3.14)$$

Given such a solution, we can switch to the point of view B), and interpret these solutions in terms of a perturbed brane  $\mathcal{M}$  with deformed geometry. We can compute its curvature by inserting the solution into the expressions (2.38) for the linearized Einstein tensor, noting the harmonic gauge<sup>12</sup> condition (2.34). This gives

$$\begin{aligned} \delta \mathcal{G}^{ab} &= \delta_\phi \mathcal{G}^{ab} + \delta_F \mathcal{G}^{ab} \\ &= -e^{-\sigma} \bar{\theta}^{ac} \bar{\theta}^{bd} \bar{K}_{cd}^A \bar{\square} \phi_A - \frac{1}{2} \bar{\theta}^{ac} \bar{G}^{bd} \bar{\square} F_{cd} - \frac{1}{2} \bar{\theta}^{bc} \bar{G}^{ad} \bar{\square} F_{cd} - \frac{1}{4} \bar{G}^{ab} (\bar{\theta}^{cd} \bar{\square} F_{cd}) \\ &= \mathcal{P}^{ab;cd} T_{cd} + \mathcal{O}(\nabla \nabla (T + T[\phi])) + \mathcal{O}((\phi, F)^2). \end{aligned} \quad (3.15)$$

The quadratic terms  $\mathcal{O}((\phi, F)^2)$  are negligible compared with the linear ones, assuming that  $K_{ab}^A$  is large for compactified branes. As explained in the next section, one can (and should) always choose a locally adapted background with  $F|_p = 0$ , hence the  $T[F]$  was dropped. The tensor

$$\begin{aligned} \mathcal{P}^{ab;cd} &= \Lambda_0^{-4} e^{-2\sigma} \bar{\theta}^{aa'} \bar{\theta}^{bb'} \bar{K}_{a'b';c'd'} \Pi_{c'e'}^{c''d''} \bar{\theta}^{c'e} \bar{\theta}^{d'e} \\ &=: G_N P_{c'd'}^{ab} \bar{G}^{c'e} \bar{G}^{d'e} \end{aligned} \quad (3.16)$$

$$G_N = \Lambda_0^{-4} r_K^{-2} \quad (3.17)$$

governs the coupling of  $T_{ab}$  to the Einstein tensor. Here

$$\bar{K}_{ab;cd} = \bar{\nabla}_a \partial_b \bar{x}_A \bar{\nabla}_c \partial_d x^A = \mathcal{O}(r_K^{-2}) \quad (3.18)$$

can be interpreted as extrinsic curvature of  $\mathcal{M} \subset \mathbb{R}^D$ , as discussed appendix A. These equations amount to modified  $2n$ -dimensional linearized Einstein equations

$$\delta \mathcal{G}^{ab} = G_N P_{cd}^{ab} T^{cd} + \mathcal{O}(\nabla \nabla (T + T[\phi])) . \quad (3.19)$$

$G_N$  plays the role of the effective  $2n$ -dimensional Newton constant, determined by the extrinsic curvature scale  $K_{ab;cd} \sim r_K^{-2}$  and the NC scale. It will reduce to  $G_N \sim \Lambda_{\text{NC}}^{-2}$  in 4 dimensions (4.49).

We will argue that the transversal perturbations  $\phi^A$  provide the leading contribution to gravity, in particular Newtonian gravity. On the other hand,  $F$  describes perturbations of the Poisson structure  $\theta^{ab}$ , which determines the local orientation of the effective frame (2.19). We can get some more insights by noting that the coupling (3.2) to matter and therefore the mixing term can be interpreted as coupling to a dipole density as in electrodynamics, determined by  $T_{ab} \theta^{bc}$  [3]. Thus the matter contribution to  $F$  can only lead to dipole and higher multipole metric deformations, and might moreover be averaged out in suitable compactifications<sup>13</sup>. In that case, GR would be recovered. On the other hand, Ricci-flatness in vacuum might be violated due to  $\delta_F \mathcal{G}^{ab} = \mathcal{O}(\nabla \nabla T[\phi]) \neq 0$  unlike in [31]. This might be relevant for "dark matter", which at present is nothing but an unexplained deviation from Ricci-flatness.

In any case we will simply omit the contributions from  $F$  in this paper, leaving a detailed investigation for future work.

<sup>12</sup>its validity in the presence of matter will be clarified later.

<sup>13</sup>For example, all Lorentz-violating tensors such as  $\langle \theta^{ab} \rangle = 0$  might be averaged out upon compactification, reducing correlators such as  $\langle P_{cd}^{ab} \rangle \neq 0$  to their 4-D Lorentz-invariant averages.

### 3.2 Locally adapted backgrounds and gravity

We now study general brane geometries at the non-linear level. The idea is to consider the space at any given point  $p \in \mathcal{M}$  as perturbation of some locally adapted, intrinsically flat (but not extrinsically flat) background  $\bar{\mathcal{M}}$  in terms of transversal and tangential perturbations

$$\begin{aligned} x^A &= \bar{x}^A + \phi^A, & \theta_{ab}^{-1} &= \bar{\theta}_{ab}^{-1} + F_{ab}, \\ \phi^A|_p &= 0 = F|_p, \end{aligned} \quad (3.20)$$

Since  $\bar{\mathcal{M}}$  is intrinsically flat,  $\theta^{ab}|_p$  can be extended on  $\bar{\mathcal{M}}$  such that

$$\bar{\nabla}^{(g)} \bar{\theta}_{ab}^{-1} = \bar{\nabla}^{(g)} \bar{G} = 0 = \partial \bar{\sigma} \quad (3.21)$$

implying that  $\bar{G}$  is covariantly constant on  $\bar{\mathcal{M}}$ , so that  $\bar{\nabla}^{(G)} = \bar{\nabla}^{(g)} \equiv \bar{\nabla}$ . It follows that the Laplacian on  $\bar{\mathcal{M}}$  is unique

$$\bar{\square} \equiv \bar{G}^{ab} \bar{\nabla}_a \bar{\nabla}_b \quad (3.22)$$

$\bar{\mathcal{M}}$  is the noncommutative analog of a "free-falling" frame, exploiting the background independence of the matrix model. However, there are many possible backgrounds  $\bar{\mathcal{M}} \subset \mathbb{R}^{10}$  which are intrinsically flat but have different extrinsic geometry: one could simply choose the tangent plane  $\bar{\mathcal{M}} = T_p \mathcal{M}$ , or one can try to match also the extrinsic curvature of  $\mathcal{M}$  with  $\bar{\mathcal{M}}$  by fitting e.g. a cylinder or a cone. The latter is clearly more appropriate, because then a linear analysis of the corresponding perturbations  $\phi_A$  suffices to compute the curvature perturbations, due to (2.28); this will become clear below. Now the Gauss-Codazzi theorem  $K_{ab;cd} - K_{bc;ad} = R_{acbd}[g]$  tells us that we cannot expect to match the extrinsic curvature completely. However, in the case of compactified extra dimensions such as  $\mathcal{M} = \mathcal{M}^4 \times T^2$ , we can require that

$$\bar{K}_{ab}^A \approx K_{ab}^A \quad \text{and} \quad \bar{K}^A = \bar{\square} \bar{x}^A = 0 \quad (3.23)$$

This implies that the intrinsic geometry of  $\mathcal{M}$  is nearly flat while the extrinsic curvature has large components, which is exactly satisfied for compactified branes.

We are now precisely in the situation of section 3.1 for linearized gravity. The perturbations  $\phi^A, F$  viewed as  $U(1)$  perturbations on  $\bar{\mathcal{M}}$  must satisfy the equations of motion (3.12), since  $\bar{\mathcal{M}}$  is intrinsically flat. We can then switch to the point of view B), and interpret these solutions in terms of a deformed brane  $\mathcal{M}$  with perturbed geometry. Its linearized Einstein tensor is obtained as in (3.15), where we assume<sup>14</sup> that  $\bar{\nabla} \bar{K}_{ab}^A = 0$  which is satisfied e.g. on cylinders. This actually computes the full Einstein tensor at  $p \in \mathcal{M}$  since  $\bar{\mathcal{M}}$  is flat. Since  $p \in \mathcal{M}$  was arbitrary, we obtain the modified  $2n$ -dimensional Einstein equations

$$\mathcal{G}^{ab} = G_N P_{cd}^{ab} T^{cd} + \mathcal{O}(\nabla \nabla (T + T[\phi])) + \mathcal{O}((\phi, F)^2). \quad (3.24)$$

where

$$\mathcal{P}^{ab;cd} = \Lambda_0^{-4} e^{-2\sigma} \theta^{aa'} \theta^{bb'} K_{a'b';c'd''} \Pi_{c'd'}^{c''d''} \theta^{c'c} \theta^{d'd} \quad (3.25)$$

$$=: G_N P_{c'd'}^{ab} G^{c'c} G^{d'd},$$

$$G_N = \Lambda_0^{-4} r_K^{-2} \quad (3.26)$$

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<sup>14</sup>cf. section 5.2 for a discussion of the general case.

Note that we tacitly replaced  $\bar{K}_{ab}^A$  by  $K_{ab}^A$ , which requires (3.23) to hold.

Let us contrast the above considerations with general relativity. The crucial difference is that  $\phi^A$  are fundamental degrees of freedom here, governed by the wave equation (3.12). Those associated with extrinsic curvature  $K_{ab}^A \neq 0$  couple linearly to  $T$  and thus mediate gravity, while the others couple to  $T$  only at the non-linear level and should therefore play a less significant role. This mechanism seems unavoidable for compactified branes in matrix models, and must play a significant role for gravity on such branes. However, this does not imply that linear perturbations give the full story, as indicated by the quadratic terms  $\mathcal{O}((\phi, F)^2)$ . At the non-linear level, there are embedding deformations called “gravity bags” in [19]. They may lead to long-range modifications<sup>15</sup> of gravity, possibly relevant to galactic or cosmological scales.

The bottom line is that the energy-momentum tensor contributes linearly to the Einstein tensor in the “semi-classical” matrix model, even without invoking any quantum corrections or induced Einstein-Hilbert terms. Transversal brane fluctuations are identified as prime carriers of gravity (possibly along with others), which in the presence of compactified branes couple linearly to matter. Ricci-flat vacuum geometries arise provided the  $\nabla\nabla T[\phi]$  contributions can be neglected, but this is not necessarily the case. This link between curvature and the e-m tensor was missing in the earlier related works [19, 25, 31], which are now part of a possibly viable mechanism of emergent gravity scenario in matrix models.

### 3.3 Gravitational excitations.

Let us briefly discuss the propagating geometrical modes (i.e. the analogs of gravitons) in the absence of matter. Although the mixing term  $S_{\text{mix}}$  prevents a full understanding at present, we can tentatively distinguish three different types of modes:

1) transversal modes  $\phi^A$  corresponding to moduli of the embedding, which couple to the extrinsic curvature and therefore to matter as discussed above. Only the massless moduli  $\phi_{(\alpha)}^A$  defined in (4.22) can contribute to long-range gravitational modes. These are quite analogous to standard gravitons (in harmonic gauge!), and indeed we obtain typically two such propagating modes, corresponding to the two embedding moduli of the compactifications  $T^2$  and  $S^3 \times S^1$ . However they will presumably mix with some  $F$  degrees of freedom in a way which remains to be clarified.

2) transversal modes  $\phi^A$  which do not correspond to extrinsic curvature directions, i.e. the remaining transversal modes of  $M^4 \times \mathcal{K} \subset \mathbb{R}^{10}$ . These are completely sterile at the linearized level, and might be viewed as massless scalar fields. However, they may become interesting at the non-linear level, describing embedding deformations called “gravity bags” in [19].

3) the  $2n - 2$  tangential modes corresponding to the “would-be gauge fields”  $F_{ab}$ . Two of these are absorbed in the scalar fields  $e^{-\sigma}$  and  $\eta \sim G^{ab}g_{ab}$  (somewhat reminiscent of dilaton and axion), and some may be contributing to the physical gravitons as pointed out above. However, their precise 4-dimensional meaning and possible relation with well-known modes such as gravitons, dilatons, or galileons [32] remains to be clarified.

Up to now, we have been studying the  $2n$ -dimensional geometry of  $\mathcal{M}$ . In order to understand the effective 4-dimensional gravity, we consider in the next section compactified

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<sup>15</sup>The analysis in [19] is based on a somewhat different scenario using a complexified Poisson structure and needs to be adapted, but qualitative features are expected to carry over to the present framework.

backgrounds in more detail. Then we will indeed obtain 4-dimensional gravity (at least for toroidal compactifications), and identify the effective 4-dimensional Newton constant.

## 4 Compactified branes and 4-dimensional gravity

We have seen that the coupling of gravity to matter requires the presence of extrinsic curvature. Let us therefore discuss in more detail branes with compactified extra dimensions

$$\mathcal{M}^{2n} = \mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^D \quad (4.1)$$

where the extrinsic curvature is predominantly due to  $\mathcal{K} \subset \mathbb{R}^D$ , while the embedding of  $\mathcal{M}^4$  is approximately flat. Such solutions for<sup>16</sup>  $\mathcal{K} = T^2$ ,  $\mathcal{K} = S^2 \times S^2$  and  $\mathcal{K} = S^3 \times S^1$  were given recently [18]. While the induced metric  $g_{ab}$  on  $\mathcal{K}$  is space-like, the effective metric  $G_{ab}$  on  $\mathcal{K}$  is degenerate or has Minkowski signature, corresponding to light-like compactification. This is possible because of "split noncommutativity", where the Poisson bi-vector relates the compact space  $\mathcal{M}^4$  with the non-compact space  $\mathcal{K}$ ,

$$\theta^{ab} \partial_a \wedge \partial_b = \theta^{\mu i}(x, y) \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial y^i} + \dots \quad (4.2)$$

where  $x^\mu$  are coordinates on  $\mathcal{M}^4$  and  $y^i$  are coordinates on  $\mathcal{K}$ . Such a structure is realized e.g. by the canonical symplectic form on the cotangent bundle  $T^*\mathcal{K}$ . If  $\mathcal{K}$  has dimension 4, then  $\mathcal{M}^4$  might even be isotropic,  $\{x^\mu, x^\nu\} = 0$ . Now recall that metric variation due to embedding fluctuations is given by  $\delta g_{ab} = -2\phi_A K_{ab}^A$  (2.28). For the present type of background, this implies that only the perturbations  $\phi_A$  of the compactification  $\mathcal{K}$  couple to matter, while the perturbations of flat  $M^4 \subset \mathbb{R}^4$  decouple. Remarkably, such perturbations of  $\mathcal{K}$  lead to perturbations of the effective 4-dimensional (!) metric on  $M^4$  due to split noncommutativity,

$$\delta_{\mathcal{K}}(\gamma^{ab} \partial_a \otimes \partial_b) \approx \theta^{ai} \theta^{bj} \delta g_{ij}^{\mathcal{K}} \partial_a \otimes \partial_b \equiv \delta \gamma_{\mathcal{K}}^{ab} \partial_a \otimes \partial_b \approx \delta \gamma_{\mathcal{K}}^{\mu\nu} \partial_\mu \otimes \partial_\nu \quad (4.3)$$

assuming  $\theta^{ij} \approx 0$ , in self-explanatory notation. This will be elaborated in detail below.

**Constant curvature compactifications and moduli.** To make this more explicit, we assume that

$$\mathcal{K} = \times_i \mathcal{K}_{(i)} \subset \mathbb{R}^D \quad (4.4)$$

is a product manifold with constant exterior curvature, in the sense that

$$\nabla_a \nabla_b \bar{x}^A = K_{ab}^A = - \sum_i \frac{1}{r_i^2} g_{ab}^{(i)} \bar{x}_{(i)}^A. \quad (4.5)$$

Here  $g_{ab}^{(i)}$  is the induced metric on  $\mathcal{K}_i$  with radius  $r_i$ . This holds e.g. for  $\mathcal{K} = T^n = \times_i S_{(i)}^1$  or  $\mathcal{K} = S^n$ , which suffices to understand the mechanism. Following the discussion in section 3.2, we choose at any given point  $p \in \mathcal{M}^4$  a locally adapted intrinsically flat background cylinder

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<sup>16</sup>The intrinsic geometry of the solutions denoted  $S^2 \times T^2$  and  $T^4$  in [18] is in fact  $S^3 \times S^1$ .



(or cone)  $T_p \mathcal{M}^4 \times \mathcal{K}$  with constant radii  $\bar{r}_i$  and  $\bar{\nabla} \bar{\theta} = 0$ . Perturbing the radii and  $F$  such that  $\delta r_i|_p = 0 = F|_p$  leads to

$$\delta g_{ab} = 2 \sum_i \frac{1}{r_i} \delta r_i g_{ab}^{(i)}, \quad (4.6)$$

where

$$\delta r_i = \frac{1}{r_i} \phi_A^{(i)} \bar{x}_{(i)}^A \quad (4.7)$$

denotes the radial moduli of the  $\mathcal{K}_i$ , which play the central role in the following. Then

$$\begin{aligned} \delta \gamma^{ab} &= 2 \sum_i \frac{1}{r_i} \delta r_i \bar{\gamma}_{(i)}^{ab} + \mathcal{O}(F) \\ \delta \sigma &= \frac{1}{n-1} \sum_i \frac{1}{r_i} \delta r_i g_{ab}^{(i)} g_{(i)}^{ab} + \mathcal{O}(F) = \frac{1}{n-1} \sum_i \dim \mathcal{K}_i \frac{1}{r_i} \delta r_i + \mathcal{O}(F) \end{aligned} \quad (4.8)$$

where

$$\gamma_{(i)}^{ab} \equiv \gamma_{\mathcal{K}_i}^{ab} = \theta^{ak} \theta^{bl} g_{kl}^{(i)}. \quad (4.9)$$

Then

$$\bar{\square} \delta \gamma^{ab} = 2 \sum_i \frac{1}{r_i} \bar{\square} \delta r_i \gamma_{(i)}^{ab} + \mathcal{O}(F). \quad (4.10)$$

using  $\bar{\nabla} \bar{\gamma}_{(i)}^{ab} = 0$  by the above assumptions. The equations of motion for these radial moduli  $\delta r_i$  are obtained from (3.12)

$$\bar{\square} \delta r_i = \frac{\Lambda_0^{-4}}{r_i} e^{-\sigma} T_{ab} \bar{\theta}^{aa'} \bar{\theta}^{bb'} \Pi_{a'b'}^{cd} g_{cd}^{(i)} = \frac{\Lambda_0^{-4}}{r_i} T_{ab} e^{-\sigma} \left( \gamma_{(i)}^{ab} - \frac{\dim \mathcal{K}_i}{2(n-1)} \gamma^{ab} \right). \quad (4.11)$$

while the story for  $F$  is as before and will not be repeated. Here  $g_{cd} g_{(i)}^{cd} = g_{cd}^{(i)} g_{(i)}^{cd} = \dim \mathcal{K}_i$ , since the embedding of  $\mathcal{K}_i$  is assumed to be orthogonal to  $\mathcal{M}^4$  (and possible other  $\mathcal{K}_j$ ). Switching to the geometrical picture, these perturbations correspond to metric perturbations with  $2n$ -dimensional Einstein tensor (2.38)

$$\begin{aligned} \mathcal{G}^{ab} &= e^{-\sigma} \sum_i \frac{1}{r_i} \bar{\gamma}_{(i)}^{ab} \bar{\square}_G \delta r_i - \frac{1}{2} \bar{G}^{bd} \bar{\theta}^{ac} \bar{\square} F_{cd} - \frac{1}{2} \bar{G}^{ad} \bar{\theta}^{bc} \bar{\square} F_{cd} + \frac{1}{4} \bar{G}^{ab} (\bar{\theta}^{cd} \bar{\square} F_{cd}) + \mathcal{O}(\delta^2) \\ &= \mathcal{P}^{ab;cd} T_{cd} + \mathcal{O}(\Lambda_0^{-4} \nabla \nabla (T + T[\phi])) + \mathcal{O}(\delta^2). \end{aligned} \quad (4.12)$$

This holds for toroidal compactifications  $M^4 \times T^{2m}$ , and  $\mathcal{O}(\delta^2)$  stands for quadratic contributions in the perturbations. Here  $\mathcal{P}$  is given by

$$\mathcal{P}^{ab;cd} = \Lambda_0^{-4} e^{-2\sigma} \sum_i \frac{1}{r_i^2} \gamma_{(i)}^{ab} \left( \gamma_{(i)}^{cd} - \frac{\dim \mathcal{K}_i}{2(n-1)} \gamma^{cd} \right). \quad (4.13)$$

## 4.1 Effective 4-dimensional gravity for toroidal compactifications

To obtain the 4-dimensional Einstein tensor, we simply perform a dimensional reduction along  $\mathcal{K}$ . This leads to the effective 4-dimensional metric as derived more generally in section 5,

$$G_{(4)}^{\mu\nu} = e^{-\sigma_4} \gamma^{\mu\nu} \quad (4.14)$$

where the normalization  $e^{-\sigma_4}$  plays the same role in 4 dimensions as  $e^{-\sigma}$  does on  $\mathcal{M}^{2n}$ . Then the  $2n$ -dimensional Laplacian  $\square$  can be related to the 4-dimensional Laplacian as follows (cf. (5.13))

$$\square_{(4)} = G_{(4)}^{\mu\nu} \nabla_\mu \nabla_\nu = e^{-\sigma_4 + \sigma} \square \quad (4.15)$$

if acting on tensor fields that are covariantly constant on  $\mathcal{K}$ . We can then repeat the above computation leading to (4.12) for  $G_{(4)}^{\mu\nu}$  on  $M^4$ , replacing  $\square_{(4)} \delta r_i$  by  $\square \delta r_i$  and subsequently using (4.11). The harmonic gauge condition still applies in 4 dimensions (5.10), and we obtain the following equation for the effective 4-dimensional Einstein tensor

$$\mathcal{G}_{(4)}^{\mu\nu} = \mathcal{P}_{(4)}^{\mu\nu;cd} T_{cd} + \mathcal{O}(\Lambda_0^{-4} \nabla \nabla T) + \mathcal{O}(\delta^2) \quad (4.16)$$

where<sup>17</sup>

$$\mathcal{P}_{(4)}^{\mu\nu;cd} = \Lambda_0^{-4} e^{-2\sigma_4} \sum_i \frac{1}{r_i^2} \gamma_{(i)}^{\mu\nu} \left( \gamma_{(i)}^{cd} - \frac{\dim \mathcal{K}_i}{2(n-1)} \gamma^{cd} \right). \quad (4.17)$$

This is similar to the 4-dimensional Einstein equations. In particular, these metrics are Ricci-flat away from matter sources, up to non-linear corrections in the strong gravity regime. We keep the  $2n$ -dimensional  $T_{ab}$  for the sake of generality, but for most applications the e-m tensor will be 4-dimensional. The effective 4-dimensional Newton constant or the Planck length is determined by the compactification scale  $r_{\mathcal{K}}^{-2}$  as well as  $\Lambda_{\text{NC}}$  analogous as in (3.26),

$$G_N = \Lambda_0^{-4} r_{\mathcal{K}}^{-2} = g_{\text{YM}}^2 e^{\sigma_4} r_{\mathcal{K}}^{-2}. \quad (4.18)$$

Here

$$\frac{1}{g_{\text{YM},4}^2} = \Lambda_0^4 e^{\sigma_4} \quad (4.19)$$

is the effective four-dimensional gauge coupling in analogy to (2.12), and  $e^{-\sigma_4}$  is related to the noncommutativity scale. This will be elaborated in more detail for  $M^4 \times T^2$  below.

We conclude that the radial moduli of  $\mathcal{K}$  describe perturbations of the effective 4-dimensional metric  $G_{(4)}^{\mu\nu}$ , encoding at least part of the gravitational degrees of freedom. Since they couple linearly to the energy-momentum tensor, Newtonian gravity is recovered, as elaborated below. This is a remarkable mechanism, which offers an unexpected solution of the moduli stabilization problem in matrix models. However, some of these moduli may become massive due to fluxes:

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<sup>17</sup>Notice that (4.12) makes sense as tensorial equation both on  $\mathcal{M}^{(2n)}$  and on  $M^4$ . However, the  $2n$ -dimensional metric does *not* preserve  $M^4$ , and any other form of (4.12) e.g. in terms of  $T^{ab}$  would not restrict to  $M^4$ .

**Flux compactifications and moduli stabilization.** In the presence of fluxes on  $\mathcal{K}$ , some of the embedding moduli of  $\mathcal{K} \subset \mathbb{R}^{10}$  are stabilized by the effective mass  $m_{AB}^2$  (3.9). To see this, recall the quadratic action (3.3) in the case of the product compactification as above. Then

$$m_{AB}^2 \phi^A \phi^B = 2 \sum_{i,j} g_{ab}^{(i)} \gamma_{(j)}^{ab} \frac{\delta r_i}{r_i} \frac{\delta r_j}{r_j}. \quad (4.20)$$

This means that the radial moduli  $\delta r_i$  are long-range propagating modes (gravitons) in the absence of flux, but may acquire a mass in the presence of a flux on  $\mathcal{K}_i$ . Note that the flux  $\theta^{ab}$  may connect different  $\mathcal{K}_i$ , which happens e.g. for fuzzy tori  $T_N^2 \sim S^1 \times S^1 \subset \mathbb{R}^4$ . In that case the mass term

$$m_{AB}^2 \phi^A \phi^B = 2\theta_{(12)}^2 \frac{\delta r_1}{r_1} \frac{\delta r_2}{r_2} \quad (4.21)$$

has indefinite signature, so that compactifications on tori with fluxes are unstable. Therefore fuzzy cylinders corresponding to tori without fluxes as studied in [18] are preferred. Then the massless modes  $\delta r_i$  are the radial modes of the cycles, which may vary along  $\mathcal{M}^4$ . More generally, we can diagonalize  $m_{AB}^2$  by some  $x$ -dependent  $SO(D-4)$  transformation,

$$m_{AB}^2 = \oplus_i m_{(i)}^2 \delta_{AB}^{(i)}. \quad (4.22)$$

The condition  $m_{(i)}^2 = 0$  determines massless moduli fields  $\delta r_i$  among the transversal perturbations  $\phi^A$ . Since they do not couple to a flux on  $\mathcal{K}$ , the fluctuations of the metric due to these massless moduli are along the non-compact  $M^4$  as in (4.3), due to split noncommutativity

$$\delta \gamma_{(i)}^{ab} \partial_a \otimes \partial_b = \delta \gamma_{(i)}^{\mu\nu} \partial_\mu \otimes \partial_\nu. \quad (4.23)$$

Note that the above mass term also applies to the nonabelian scalars. Therefore a flux on  $\mathcal{K}$  typically leads to SUSY breaking, which is well-known in string theory (see e.g. [28]). If  $\mathcal{K}$  is 4-dimensional, then there should be no flux on  $\mathcal{K}$  if the two transversal degrees of freedom are to remain massless. These issues are discussed further in section 5.

## 4.2 Explicit example: $M^4 \times T^2$

Let us work out the above results explicitly for the case of compactifications on  $T^2$ ,

$$\mathcal{M} = \mathbb{R}^4 \times T^2 \subset \mathbb{R}^D. \quad (4.24)$$

We can assume that the non-compact  $\mathbb{R}^4$  is embedded along the 0, 1, 2, 3 directions. To obtain  $\mathbb{R}^4 \times T^2 \subset \mathbb{R}^8$ , we start with two fuzzy cylinders  $(U_4, X^2)$  and  $(U_5, X^3)$  with NC parameter  $\kappa_{(4,5)}$  and radii  $r_{4,5}$  defined via

$$\begin{aligned} [U_4, X^2] &= \kappa_4 U_4, & [U_5, X^3] &= \kappa_5 U_5, \\ U_i U_i^\dagger &= r_i^2, & i &= 4, 5. \end{aligned} \quad (4.25)$$

We can make them rotate along a two-dimensional non-commutative plane  $[X^\mu, X^\nu] = i\theta^{\mu\nu}$ ,  $\mu = 0, 1$  (which commutes with the cylinders) as follows [18]

$$X^A = \begin{pmatrix} X^{0,1} \\ X^2 \\ X^3 \\ X^4 + iX^5 \\ X^6 + iX^7 \end{pmatrix} = \begin{pmatrix} X^\mu \\ U_4 e^{ik_\mu^{(4)} X^\mu} \\ U_5 e^{ik_\mu^{(5)} X^\mu} \end{pmatrix}. \quad (4.26)$$

These are solution of the matrix equations of motion

$$\square X^A = 0 \quad \text{if} \quad k_\mu^{(i)} k_\nu^{(i)} \theta^{\mu\mu'} \theta^{\nu\nu'} \eta_{\mu'\nu'} = -\kappa_i^2 \quad (\text{no sum over } i), \quad (4.27)$$

provided  $[k_\mu^{(4)} X^\mu, k_\nu^{(5)} X^\nu] = 0$ . These solutions describe  $\mathbb{R}^4 \times T^2$  where the torus rotates along the non-compact space, and is stabilized by angular momentum. Note that the only non-vanishing 4-dimensional component of  $\theta^{\mu\nu}$  is  $\theta^{01} \neq 0$ , where  $x^0$  is time-like w.r.t.  $g_{ab}$ . This is essential to obtain an effective 4-dimensional metric with Minkowski signature, as we will see. Moreover, this 6-dimensional solution behaves as a 4-dimensional (!) space  $\mathbb{R}^2 \times T^2$  in the UV [18], such that the IKKT model is (expected to be) perturbatively finite on this background.

**Semi-classical analysis.** To gain more insights into this solution and its effective metric, we consider the semi-classical limit. Then the above solution can be described in terms of a 6-dimensional plane compactified on a 2-torus. Consider 6-dimensional coordinates  $\xi^a = (x^\mu, \xi^4, \xi^5)$ , and  $U_4 \sim r_4 e^{i\zeta^4}$  and  $U_5 \sim r_5 e^{i\zeta^5}$ . Then (4.26) can be written in a compact way as

$$x^A = \begin{pmatrix} x^\mu \\ r_4 \exp(i(k_\mu^{(4)} x^\mu + \zeta^4)) \\ r_5 \exp(i(k_\mu^{(5)} x^\mu + \zeta^5)) \end{pmatrix} \equiv \begin{pmatrix} x^\mu \\ r_4 \exp(i\xi^4) \\ r_5 \exp(i\xi^5) \end{pmatrix} \quad (4.28)$$

(dropping constant phase shifts). The tori are compactified along the 6-dimensional momenta

$$k^{(4)} = (k_{(4)}^\mu, 1, 0), \quad k^{(5)} = (k_{(5)}^\mu, 0, 1) \quad (4.29)$$

in the  $(x^\mu, \zeta^i)$  coordinates, or along  $\xi^4, \xi^5$  in the  $\xi^a$  coordinates; the latter are more useful here. The Poisson tensor can be written as

$$\theta^{ab} = \{\xi^a, \xi^b\} = \theta \begin{pmatrix} 0 & c & 0 & 0 & \vartheta_4^0 & \vartheta_5^0 \\ -c & 0 & 0 & 0 & \vartheta_4^1 & \vartheta_5^1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\vartheta_4^0 & -\vartheta_4^1 & -1 & 0 & 0 & 0 \\ -\vartheta_5^0 & -\vartheta_5^1 & 0 & -1 & 0 & 0 \end{pmatrix}. \quad (4.30)$$

Of course we could admit more general  $\vartheta_{4,5}^\mu$ . Here the no-flux condition  $\theta^{45} = 0$  is already imposed, which amounts to<sup>18</sup>

$$[z^4, z^5] = 0, \quad z^4 = x^4 + ix^5, \quad z^5 = x^6 + ix^7. \quad (4.31)$$

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<sup>18</sup>this leads to a constraint on  $k_3^{(4)}$  and  $k_2^{(5)}$  in (4.28).

The embedding metric is obviously flat, given by

$$g_{ab} = (\eta_{\mu\nu}, 4\pi^2 r_i^2 \delta_{ij}) = \text{diag}(-1, 1, 1, 1, 4\pi^2 r_4^2, 4\pi^2 r_5^2) \quad (4.32)$$

which is diagonal in the  $\xi^a$  coordinates. Therefore  $e^{(a)} = \theta^{ab} \partial_b$  defines a frame (2.19) of orthogonal (but not orthonormal) tangent vectors on  $\mathbb{R}^4 \times T^2$ , which satisfy

$$(e^{(a)}, e^{(b)})_G = \theta^{aa'} \theta^{bb'} G_{a'b'} = e^\sigma g^{ab} = e^\sigma (\eta^{\mu\nu}, \frac{1}{4\pi^2 r_i^2} \delta_{ij}). \quad (4.33)$$

Note that the effectively time-like vector  $e^{(0)} = c\theta\partial_1 + \dots$  is pointing along  $x^1$  (rather than  $x^0$ !) and is wrapping the torus. The 6-dimensional conformal metric is given in  $\xi^a$  coordinates by

$$\gamma^{ab} = \theta^{aa'} \theta^{bb'} g_{a'b'} = \theta^2 \left( \begin{array}{ccc|ccc} & & & & -c\vartheta_4^1 & -c\vartheta_5^1 \\ & & & & -c\vartheta_4^0 & -c\vartheta_5^0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline \theta^{-2}\gamma_{(4-d)}^{\mu\nu} & & & & & \\ \hline -c\vartheta_4^1 & -c\vartheta_4^0 & 0 & 0 & 1 - (\vartheta_4^0)^2 + (\vartheta_4^1)^2 & -\vartheta_4^0\vartheta_5^0 + \vartheta_4^1\vartheta_5^1 \\ -c\vartheta_5^1 & -c\vartheta_5^0 & 0 & 0 & -\vartheta_4^0\vartheta_5^0 + \vartheta_4^1\vartheta_5^1 & 1 - (\vartheta_5^0)^2 + (\vartheta_5^1)^2 \end{array} \right).$$

In particular the metric  $G^{ab}$  of  $\mathbb{R}^4 \times T^2$  is also flat, but it is not a product metric:  $T^2$  is not perpendicular to  $\mathbb{R}^4$ . This must be so, because the compactification must be time-like *and* the 4-dimensional space must have Minkowski signature. However, all we need for 4-dimensional physics is the 4-dimensional metric, which is given by

$$\begin{aligned} G_{(4)}^{\mu\nu} &= e^{\sigma_4} \gamma_{(4-d)}^{\mu\nu} \equiv e^{\sigma_4} \gamma^{\mu\nu}, \\ \gamma^{\mu\nu} &= \theta^{\mu a} \theta^{\nu b} g_{ab} = \theta^2 \text{diag}(c^2, -c^2, 0, 0) + e^{(4)\mu} e^{(4)\nu} 4\pi^2 r_4^2 + e^{(5)\mu} e^{(5)\nu} 4\pi^2 r_5^2 \\ &= -e^{(0)\mu} e^{(0)\nu} c^2 + e^{(1)\mu} e^{(1)\nu} c^2 + e^{(4)\mu} e^{(4)\nu} 4\pi^2 r_4^2 + e^{(5)\mu} e^{(5)\nu} 4\pi^2 r_5^2 \end{aligned} \quad (4.34)$$

which is clearly non-degenerate with Minkowski signature. The  $e^{(i)}$  for  $i = 0, 1, 4, 5$  form an orthogonal (but not orthonormal) frame for the effective 4-dimensional metric, where  $e^{(0)}$  is time-like and the others are space-like. Explicitly the 4-dimensional frame is

$$\begin{aligned} e^{(4)\mu} &= \theta(\vartheta_4^\mu + \delta_2^\mu), & e^{(1)\mu} &= \theta\delta_0^\mu \\ e^{(5)\mu} &= \theta(\vartheta_5^\mu + \delta_3^\mu), & e^{(0)\mu} &= \theta\delta_1^\mu \end{aligned} \quad (4.35)$$

where  $e^{(4,5)a} \neq 0$  only for 4-dimensional indices  $a \rightarrow \mu = 0, 1, 2, 3$ . The equations of motion  $\square z^j = 0$  reduce to

$$\begin{aligned} 0 = G^{44} &\propto 1 - (\vartheta_4^0)^2 + (\vartheta_4^1)^2, \\ 0 = G^{55} &\propto 1 - (\vartheta_5^0)^2 + (\vartheta_5^1)^2, \end{aligned} \quad (4.36)$$

expressing the fact that the compactification is light-like. A possible solution is

$$\begin{aligned} \vartheta_4^0 &= 1 = \vartheta_5^0, & \vartheta_4^1 &= 0 = \vartheta_5^1 \\ e^{(4)\mu} &= \theta(1, 0, 1, 0), & e^{(5)\mu} &= \theta(1, 0, 0, 1). \end{aligned} \quad (4.37)$$

The conformal factors are determined by (5.4)

$$\begin{aligned} e^{-\sigma} \sqrt{|G_{ab}|} &= \sqrt{|\theta_{ab}^{-1}|} = \frac{e^{-\sigma_4}}{V_0} \sqrt{|G_{\mu\nu}^{(4)}|}, \\ e^{\sigma_4} &= V_0 \sqrt{|\theta_{ab}^{-1}| |\gamma_{\mu\nu}^{(4)}|^{-1}} \end{aligned} \quad (4.38)$$

where  $V_0 = \int_{T^2} d\xi^4 d\xi^5 = (2\pi)^2$  in the  $\xi^a$  coordinates.

**Metric fluctuations.** Now we can illustrate the mechanism for gravity. Consider transversal fluctuations  $\phi^A = \delta x^A$  around the above toroidal background  $z^i = r_i \exp i\xi_i$ ,  $i = 4, 5$  as in section 2.2.1. We will use a linearized approach here, avoiding to use locally adapted cones, and tangential perturbations  $F$  will be neglected. Then the transversality constraint (2.21) is identically satisfied by the ansatz

$$\delta z^i = \phi^i = \frac{z^i}{r_i} \delta r_i, \quad i = 4, 5 \quad (4.39)$$

in complex notation. These radial fluctuations  $\delta r_i$  lead to metric fluctuations

$$\begin{aligned} \delta g_{ii} &= -2\phi_A K_{ii}^A = 8\pi^2 r_i \delta r_i, \quad i = 4, 5 \quad (\text{no sum}), \\ \delta \gamma^{ab} &\rightarrow \delta \gamma^{\mu\nu} = \sum_{i=4,5} \delta \gamma_{(i)}^{\mu\nu} = 8\pi^2 \left( e_{(4)}^\mu e_{(4)}^\nu r_4 \delta r_4 + e_{(5)}^\mu e_{(5)}^\nu r_5 \delta r_5 \right) \\ \delta G^{ab} &= e^{-\sigma} \delta \gamma^{ab} - G^{ab} \delta \sigma \end{aligned} \quad (4.40)$$

where

$$\delta \sigma = \frac{1}{4} g^{ab} \delta g_{ab} = \frac{1}{2} (r_4^{-1} \delta r_4 + r_5^{-1} \delta r_5) \quad (4.41)$$

Note again that  $\delta \gamma^{ab} \partial_a \otimes \partial_b = \delta \gamma^{\mu\nu} \partial_\mu \otimes \partial_\nu$  affects only the 4-dimensional metric on  $\mathbb{R}^4$ , in accordance with (4.3). Hence the  $\delta r_i$  become two space-like metric degrees of freedom (4.34), governed by the effective action (3.8), (3.2)

$$\begin{aligned} S &\sim \int_{\mathcal{M}^6} \sqrt{|G|} \left( \Lambda_0^4 G^{ab} \partial_a \phi^A \partial_b \phi_A + \frac{1}{2} T_{ab} \delta G^{ab} \right) \\ &= \int_{M^4} \sqrt{|G_{(4)}|} \left( \Lambda_0^4 \sum_{i=4,5} G_{(4)}^{\mu\nu} \partial_\mu \delta r_i \partial_\nu \delta r_i + T_{\mu\nu} e_{(i)}^\mu e_{(i)}^\nu 4\pi^2 e^{-\sigma_4} r_i \delta r_i - \frac{1}{2} T_{\mu\nu} \gamma^{\mu\nu} e^{-\sigma_4} r_i^{-1} \delta r_i \right) \\ &= \int_{M^4} \sqrt{|G_{(4)}|} \left( \Lambda_0^4 \sum_{i=4,5} G_{(4)}^{\mu\nu} \partial_\mu \delta r_i \partial_\nu \delta r_i + \frac{1}{2} T_{\mu\nu} \delta G_{(4)}^{\mu\nu} \right) \end{aligned} \quad (4.42)$$

dropping the mixing contributions  $S_{\text{mix}}$  for simplicity. Here we note that

$$G^{ab} \partial_a \delta r (z^i, \partial_b z^i)_g = 0 \quad (4.43)$$

using the on-shell condition  $G^{44} = G^{55} = 0$ , and furthermore  $m_{AB}^2 = 0$  since there is no flux on  $T^2$ . We restrict ourselves to the lowest KK modes  $\delta r_i = \delta r_i(x^\mu)$ , and assume  $T^{\mu\nu}$  is 4-dimensional. The perturbation of the effective 4-dimensional metric is<sup>19</sup>

$$\begin{aligned} \delta G_{(4)}^{\mu\nu} &= e^{-\sigma_4} \delta \gamma^{\mu\nu} - G_{(4)}^{\mu\nu} \delta \sigma_4 \\ \delta \sigma_4 &= \frac{1}{2} \gamma_{\mu\nu} \delta \gamma^{\mu\nu} = r_4^{-1} \delta r_4 + r_5^{-1} \delta r_5 \end{aligned} \quad (4.44)$$

using (4.38). Therefore we obtain the equations of motion for  $\delta r_i$

$$\Lambda_0^4 \square_{(4)} \delta r_i = T_{\mu\nu} e_{(i)}^\mu e_{(i)}^\nu 4\pi^2 e^{-\sigma_4} r_i - T_{\mu\nu} \gamma^{\mu\nu} e^{-\sigma_4} \frac{1}{2r_i} = T_{\mu\nu} e^{-\sigma_4} \frac{1}{r_i} \left( \gamma_{(i)}^{\mu\nu} - \frac{1}{2} \gamma^{\rho\eta} \right). \quad (4.45)$$

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<sup>19</sup>The attentive reader may notice an apparent mismatch between the trace contributions from the 2n- and the 4-dimensional point of view in (4.42), which will be resolved in the next section.

Note that the  $\delta r_i(x)$  are indeed massless moduli, reflecting the fact that the on-shell condition (4.36) is independent of the radii  $r_i$ . This leads to the following equation for the 4-dimensional linearized Einstein tensor

$$\delta_\phi \mathcal{G}_{(4)}^{\mu\nu} = e^{-\sigma_4} \sum_i \frac{1}{r_i} \gamma_{(i)}^{\mu\nu} \square_{(4)} \delta r_i = \mathcal{P}_{(4)}^{\mu\nu;\rho\sigma} T_{\rho\sigma} \quad (4.46)$$

in agreement with (4.16), setting  $F = 0$  and taking into account (5.6). Here  $\mathcal{P}$  is given explicitly by

$$\mathcal{P}_{(4)}^{\mu\nu;\rho\sigma} = \Lambda_0^{-4} e^{-2\sigma_4} \sum_i \frac{1}{r_i^2} \gamma_{(i)}^{\mu\nu} \left( \gamma_{(i)}^{\rho\sigma} - \frac{1}{2} \gamma^{\rho\sigma} \right). \quad (4.47)$$

For  $T^{\mu\nu} = 0$ , we obtain indeed 2 propagating gravitational degrees of freedom encoded in  $\delta r_4$  and  $\delta r_5$ . The coupling to matter will be studied next. To simplify the expressions, we will set  $r_4 = r_5 = \frac{c}{2\pi} \equiv r_K$  from now on, so that the NC scale is defined appropriately as  $\Lambda_{\text{NC}}^{-2} = 2\pi r_K \theta$  in the parametrization (4.30). Then we find from (4.38)

$$V_0 \sqrt{|\theta^{ab}|} = \Lambda_{\text{NC}}^{-6} r_K^{-2}, \quad \sqrt{|\gamma_{(4-d)}^{\mu\nu}|} = \Lambda_{\text{NC}}^{-8}, \quad e^{\sigma_4} = \Lambda_{\text{NC}}^{-2} r_K^2 \quad (4.48)$$

so that the Newton constant is obtained as in (4.18)

$$G_N = 2\Lambda_0^{-4} r_K^{-2} = 2e^{\sigma_4} g_{\text{YM}}^2 r_K^{-2} = 2g_{\text{YM}}^2 \Lambda_{\text{NC}}^{-2} \quad (4.49)$$

up to factors of order 1. Remarkably, the compactification radius  $r_K$  drops out, and the 4-dimensional Planck scale is set by the noncommutativity scale and the gauge coupling constant.

**Gravitational field of a point particle.** Now consider a point mass  $m$  on the above background, moving along a time-like straight trajectory  $v$ , with a localized energy-momentum tensor  $T_{\mu\nu} \propto m v_\mu v_\nu$  and  $v^2 = G_{(4)}^{\mu\nu} v_\mu v_\nu = -1$ . We can go to coordinates where the particle is at rest located at  $\vec{x} = 0$ . To obtain the effective metric perturbation caused by  $m$ , we solve equation (4.45)

$$\square_{(4)} \delta r_i = \Delta_{(3)} \delta r_i = m \Lambda_0^{-4} \frac{1}{r_K} \left( v_{(i)}^2 + \frac{1}{4} \right) \delta^{(3)}(\vec{x}), \quad v_{(i)} = (e_{(i)}, v)_G. \quad (4.50)$$

For simplicity we assume that  $v_{(2)}, v_{(3)} \approx 0$ . Then the solution is

$$\frac{\delta r_i(x)}{r_K} = -e^{\sigma_4} g_{\text{YM}}^2 \frac{1}{r_K^2} \frac{m}{|\vec{x}|} = -\frac{1}{2} G_N \frac{m}{|\vec{x}|} \quad (4.51)$$

This means that the radius of the torus decreases in the presence of a mass, and the classical Schwarzschild radius corresponds to  $\delta r_i = -r_K$  i.e.  $r_i \approx 0$ . The resulting 4-dimensional metric perturbation is obtained as

$$\begin{aligned} \delta G_{(4)}^{\mu\nu} &= e^{-\sigma_4} \delta \gamma^{\mu\nu} - G_{(4)}^{\mu\nu} \delta \sigma_4 = 2e^{-\sigma_4} \frac{1}{r_K} \sum_i \gamma_{(i)}^{\mu\nu} \delta r_i - \frac{1}{r_K} G_{(4)}^{\mu\nu} \left( \sum_i \delta r_i \right) \\ \delta G_{\mu\nu}^{(4)} &= -2e^{-\sigma_4} \frac{1}{r_K} \sum_i G_{\mu\mu'}^{(4)} G_{\nu\nu'}^{(4)} \gamma_{(i)}^{\mu'\nu'} \delta r_i + \frac{1}{r_K} G_{\mu\nu}^{(4)} \left( \sum_i \delta r_i \right) \end{aligned} \quad (4.52)$$

recalling that  $\delta G_{\mu\nu} = -G_{\mu\mu'} G_{\nu\nu'} \delta G^{\mu\nu}$ . Therefore an observer at rest with respect to the source particle  $m$ , thus with velocity  $v$ , feels a static gravitational potential given by

$$\begin{aligned} V(x) &= -v^\mu v^\nu \delta G_{\mu\nu}^{(4)} = -2m e^{\sigma_4} \frac{1}{r_K^2} (v_{(4)}^2 + v_{(5)}^2 - v^2) \frac{1}{|\vec{x}|} \\ &\approx -G_N \frac{m}{|\vec{x}|}, \quad G_N = 2g_{\text{YM}}^2 \Lambda_{\text{NC}}^{-2} \end{aligned} \quad (4.53)$$

as long as  $v_{(i)}^2 \ll 1$ , using (4.48). This is indeed the attractive gravitational potential of a point mass, with Newton constant (4.49). Notice that the main contribution for this potential comes from the trace contribution in (4.47). However the  $F$  contribution is neglected here, and a more complete treatment will be given elsewhere.

Comparing (4.46) with the Einstein equations, we can consider the rhs as an effective modified energy-momentum tensor  $\mathcal{P}T$ . In the above example, it implies that the point mass behaves like a particle with anisotropic pressure in general relativity. While the weak equivalence principle (universality of the metric) essentially holds<sup>20</sup>, the strong equivalence principle is clearly violated, because the tensor  $\mathcal{P}$  is not Lorentz invariant in the present compactification. Spherical symmetry and isotropy might be restored either in more sophisticated compactifications as discussed in the next section, or possibly via the contributions from  $F$ . We can also verify the weak energy condition for  $\mathcal{P}T$ , which requires that

$$v'_\mu v'_\nu (\mathcal{P}T)^{\mu\nu} \geq 0 \quad (4.54)$$

for any time-like vector  $v'$  on  $M^4$ . This is indeed satisfied,

$$\begin{aligned} v'_\mu v'_\nu (\mathcal{P}T)^{\mu\nu} &= m \Lambda_0^{-4} \frac{1}{r_K^2} e^{-\sigma_4} \sum_i v'_\mu v'_\nu \gamma_{(i)}^{\mu\nu} \left( v_{(i)}^2 + \frac{1}{2} \right) \\ &= m \Lambda_0^{-4} \frac{1}{r_K^2} \sum_i (v')_{(i)}^2 \left( v_{(i)}^2 + \frac{1}{2} \right) \geq 0. \end{aligned} \quad (4.55)$$

### 4.3 Discussion

Let us discuss some aspects of the resulting gravity theory. The basic question is if realistic (linearized) gravity can be recovered along these lines. In the simplest version with toroidal compactification,  $\mathcal{P}$  is anisotropic, and only certain components of  $T^{\mu\nu}$  couple to gravity. However, this might be fixed in various ways, such as a more sophisticated compactification, or by taking the  $F$  contributions and their mixing with the  $\phi^A$  properly into account.

It is interesting to note that the 4-dimensional Planck scale  $\Lambda_{\text{planck}}^2 \sim G_N^{-1} \sim g_{\text{YM}}^{-2} \Lambda_{\text{NC}}^2$  is indeed determined by the scale of noncommutativity, as may have been expected on naive grounds. In particular, the weakness of gravity arises as a natural self-consistency condition for the semi-classical compactified geometry. Notice also that no cosmological constant arises in (4.16). This does not rule out however possible cosmological solutions with a similar behavior. Indeed the mechanism also applies if  $\mathcal{M}^4 \subset \mathbb{R}^{10}$  has extrinsic curvature, which may be interesting in the context of cosmology, as illustrated<sup>21</sup> in [19, 29]. The role of quantum fluctuations will be discussed below.

<sup>20</sup>There might be slight violations due to the dilaton or the non-standard spin connection for fermions.

<sup>21</sup>These solutions assume a certain complexification of the Poisson structure which we do not adopt here. However it seems plausible that similar types of solutions exist also in the present setup.



**Structural aspects.** Some structural remarks are in order. First, one might worry that the lack of manifest diffeomorphism invariance of the matrix model leads to inconsistencies such as ghosts. This is not the case. The simplest way to see this is to view the same model locally as  $U(1)$  NC Yang-Mills theory on  $\mathbb{R}^{2n}$ . Then there are massless propagating gauge and scalar fields after performing the usual gauge fixing procedure, and consistency is manifest. From the geometric point of view, the point is that the metric fluctuations are automatically in harmonic gauge (2.34) in the matrix coordinates, so that they are physical apart from the pure gauge contributions corresponding to symplectomorphisms or would-be  $U(1)$  gauge transformations, which are not part of the physical Hilbert space.

Let us discuss the geometrical degrees of freedom in more detail. There are  $10 - 2 = 8$  physical degrees of freedom on  $\mathcal{M}^{2n} \subset \mathbb{R}^{10}$  in the  $U(1)$  sector of the IKKT model, taking into account gauge invariance. They split into  $2n - 2$  tangential and  $10 - 2n$  transversal degrees of freedom. We have seen that the transversal moduli  $\phi^A$  of  $\mathcal{K}$  are clearly gravitational modes. The remaining 2 transversal modes (in the case of  $2n = 6$ ) may be interpreted as sterile scalar fields, which may lead to non-standard gravitational effects such as gravity bags [19]. This seems at odds with previous proposals [3, 30, 31] to interpret the two propagating  $A_\mu$  on  $M^4$  in terms of gravitons. However, two of these tangential degrees of freedom may be absorbed in the two scalar fields  $e^{-\sigma}$  and  $\eta \sim G^{\mu\nu} g_{\mu\nu}$  corresponding to dilaton and axion, and others may be frozen by the flux condition for  $\mathcal{K}$ . In the presence of matter, these tangential perturbations  $F$  may contribute to the 6 geometrical degrees of freedom required for the most general 4-dimensional effective metric.

Note also that the vacua under consideration break the global  $SO(6)$  symmetry of the model, and can be viewed in terms of time-dependent VEV's of scalar fields from a 4-dimensional NC field theory perspective. Accordingly, some of the massless would-be  $U(1)$  modes can be viewed as Goldstone bosons resulting from the breaking of the global  $SO(6)$  (or even  $SO(9, 1)$ ) symmetry of the model by the background, although this analogy goes somewhat beyond<sup>22</sup> the standard field-theoretical setting due to the presence of  $\theta^{\mu i}$ .

**Relation with string theory.** From a string theory point of view, it is remarkable that the effective gravity is indeed 4-dimensional, even though the brane  $\mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^{10}$  is embedded in a higher-dimensional non-compact target space. This is in contrast to the conventional picture where gravity originates from closed strings which propagate in 10 dimensions, leading to a 10-dimensional Newton law if embedded in  $\mathbb{R}^{10}$ . The crucial point here is that the effective brane gravity is governed by the open string metric which encodes a non-degenerate  $B$ -field, realizing split noncommutativity  $\theta^{\mu i} \neq 0$  and large extrinsic<sup>23</sup> curvature of  $\mathcal{K} \subset \mathbb{R}^{10}$ . Then the compactification moduli couple appropriately to 4-dimensional matter and mediate brane gravity. In contrast, the bulk gravity arises in a holographic manner. This origin for a 4-dimensional behavior is very different from e.g. the DGP mechanism [22], which is based on a combination of brane and bulk physics with Einstein-Hilbert term but without a  $B$  field.

The fact that 4-dimensional gravity can arise on branes in a non-compact bulk is very interesting. It means that there is no need to consider the vast landscape of 6-dimensional string compactifications with its inherent lack of predictivity. Rather, there is a mini-landscape

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<sup>22</sup>cf. also [33] for a somewhat related recent discussion.

<sup>23</sup>In particular, an abstract DBI-type action for the brane would not reproduce the above mechanism unless the induced "closed string" metric on the brane is properly realized as an embedding metric.

of at most 4-dimensional compactifications  $\mathcal{M}^{2n} = \mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^{10}$ , which is not only much smaller but also governed by a meaningful selection mechanism given by the matrix model. In principle one can even put the model on a computer, which has recently provided interesting evidence in favor of effectively 4-dimensional vacuum geometries [34].

Some remarks on the claimed UV finiteness are in order. This claim is based on two grounds: 1) the compactified brane backgrounds  $M^4 \times \mathcal{K} \subset \mathbb{R}^{10}$  behave in the UV as 4-dimensional noncommutative spaces [18], and 2) the IKKT model is equivalent to  $\mathcal{N} = 4$  NC SYM on a 4-dimensional background, and thus free of pathological UV/IR mixing and perturbatively finite, cf. [35]. Point 1) is very intuitive, since compact NC spaces can carry only finitely many degrees of freedom, and shown explicitly in [18]. Point 2) needs to be confirmed more rigorously, but is very reasonable. Note that we consider the matrix model as fundamental and independent from string theory, hence there are no other degrees of freedom apart from the ones captured by NC gauge theory. This may deviate from string theory, which contains also an infinite tower of closed string modes whose relation to the matrix model is unclear; the relation with NC field theory is established only in a suitable  $\alpha' \rightarrow 0$  limit [4]. In fact there is no claim for perturbative finiteness in the matrix model for genuinely higher-dimensional backgrounds such as  $\mathbb{R}_\theta^6$  or  $\mathbb{R}_\theta^{10}$ , as already pointed out in [1]. Therefore the present claim to obtain a (perturbatively) UV finite theory including gravity is based on the special case of compactified 4-dimensional brane solutions, and is independent of the UV finiteness of string theory. This may even be viewed as an argument in favor of 4-dimensional branes in the IKKT model.

**Vacuum energy and the cosmological constant problem.** It is remarkable that the geometric equations (4.16) resp. (4.46) do not involve any cosmological constant. However, the energy-momentum tensor might of course contain a component  $T_{\mu\nu} \propto G_{\mu\nu}$  induced by quantum fluctuations, which typically happens upon quantization. Since the model is supersymmetric, only modes below the SUSY breaking scale  $\Lambda_{\text{SUSY}}$  contribute, so that  $T_{\mu\nu}^{(\text{vac})} \sim \Lambda_{\text{SUSY}}^4 G_{\mu\nu}$ . This would modify equation (4.16) with a cosmological constant term similarly as in GR, and it appears that the usual cosmological constant problem would arise. However, this conclusion is premature. The structure of the compactified vacuum geometry  $\mathcal{M}^{2n} = \mathcal{M}^4 \times \mathcal{K}$  must be determined by taking into account all contributions to the effective action, including such quantum effects. While this is beyond the scope of the present paper, we can give a simple argument in favor of solutions with flat 4-dimensional geometry  $\mathcal{M}^4$  and constant compactification, even in the presence of vacuum energy. To this end, note that the semi-classical action (2.16)  $S \sim \Lambda_0^4 \int \sqrt{G} G^{ab} g_{ab}$  has a similar structure as the induced vacuum action  $S_{\text{vac}} \sim \Lambda_{\text{SUSY}}^4 \int \sqrt{G}$ . It is then easy to see that the equations of motion obtained from the combined action  $S = \int \sqrt{G} (\Lambda_0^4 G^{ab} g_{ab} + \Lambda_{\text{SUSY}}^4)$  take the form

$$\square_{\tilde{G}} x^a = 0, \quad \tilde{G}^{ab} = G^{ab} + \alpha \frac{\Lambda_{\text{SUSY}}^4}{\Lambda_0^4} g^{ab}. \quad (4.56)$$

This has the same structure as the bare e.o.m. used throughout this paper, with a small modification of the effective metric suppressed by  $\frac{\Lambda_{\text{SUSY}}^4}{\Lambda_0^4}$ . It is therefore very plausible that the main results of this paper apply also upon taking into account vacuum energy, and at least nearly-flat vacuum geometries should exist even in the presence of vacuum energy. In a similar vein, the full Dyson-Schwinger equations of the quantized matrix model take the

form of the bare matrix model equations  $\langle \square X^A \rangle = 0$  (dropping fermions). These arguments strongly suggest that the cosmological constant problem may be less serious or even resolved in the present approach. However this requires a careful study of the model at the quantum level, which is beyond the present paper.

## 5 More general compactifications

### 5.1 Effective 4-dimensional metric and averaging

One problem of the above background is that the radial moduli correspond to specific metric degrees of freedom, which couple to the energy-momentum tensor via  $\mathcal{P}$  in an anisotropic way. This problem may be alleviated if the metric components  $\gamma_{(i)}^{\mu\nu}$  are not constant along  $\mathcal{K}$  (or  $\mathcal{M}$ ) but rotate along  $\mathcal{K}$ , thus probing more degrees of freedom of  $T^{\mu\nu}$ . For example, this is expected to happen for a compactification on  $S^3 \times S^1$ .

To understand the effective 4-dimensional metric on more general compactified backgrounds  $\mathcal{M}^4 \times \mathcal{K}$ , we can decompose the fields in harmonics i.e. Kaluza-Klein modes on  $\mathcal{K}$ , and restrict ourselves to the lowest KK mode. For example, consider a scalar field  $\varphi = \varphi(x^\mu)$  which is constant along the compact space  $\mathcal{K}$ . Then the action in the M.M. reduces to

$$S[\varphi] = \int d^{2n}x \sqrt{|G|} G^{ab} \partial_a \varphi \partial_b \varphi = \int d^4x \left( \int_{\mathcal{K}} d\zeta \sqrt{\det \theta_{ab}^{-1} \gamma^{\mu\nu}} \right) \partial_\mu \varphi \partial_\nu \varphi. \quad (5.1)$$

Assuming that  $\det \theta_{ab}^{-1}$  is constant along  $\mathcal{K}$ , we define a reference volume  $V_0$  of  $\mathcal{K}$  via<sup>24</sup>

$$\int d^{2n}x \sqrt{\det \theta_{ab}^{-1}} = \int d^4x \int_{\mathcal{K}} d\zeta \sqrt{\det \theta_{ab}^{-1}} =: \int d^4x V_0 \sqrt{\det \theta_{ab}^{-1}} \quad (5.2)$$

as anticipated previously. We do not require that  $\gamma^{\mu\nu} = \theta^{\mu a} \theta^{\nu b} g_{ab}$  is constant along  $\mathcal{K}$ . Then the effective 4-dimensional metric is determined by the reduced  $2n$ -dimensional conformal metric averaged over  $\mathcal{K}$ . The appropriate scale factor of the effective 4-dimensional metric is determined as in (2.11) such that

$$S[\varphi] = \int d^4x \sqrt{|G_{(4)}|} G_{(4)}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi, \quad (5.3)$$

which leads to

$$G_{(4)}^{\mu\nu} = e^{-\sigma_4} \bar{\gamma}^{\mu\nu}, \quad \bar{\gamma}^{\mu\nu} = \int_{\mathcal{K}} \frac{d\zeta}{V_0} \gamma^{\mu\nu}, \quad e^{-\sigma_4} = \frac{V_0 \sqrt{|\theta_{ab}^{-1}|}}{\sqrt{|G_{\mu\nu}^{(4)}|}}. \quad (5.4)$$

Therefore  $G_{(4)}^{\mu\nu}$  is the effective metric which governs the 4-dimensional physics of the lowest KK modes. It follows as in (4.44) that

$$\delta G_{(4)}^{\mu\nu} = e^{-\sigma_4} \delta \bar{\gamma}^{\mu\nu} - G_{(4)}^{\mu\nu} \delta \sigma_4, \quad \delta \sigma_4 = \frac{1}{2} \bar{\gamma}_{\mu\nu} \delta \bar{\gamma}^{\mu\nu}. \quad (5.5)$$

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<sup>24</sup>Assuming a product structure  $\mathcal{M}^4 \times \mathcal{K}$ , one can integrate the symplectic measure over  $\mathcal{K}$  and obtain a volume form on  $\mathcal{M}^4$ . We assume that this can be done via e.g. some fixed canonical invariant measure on a homogeneous space indicated by  $\int_{\mathcal{K}} d\zeta$ .

The coupling to matter can be written for the lowest Kaluza-Klein modes either in  $2n$  or in 4 dimensional form,

$$\begin{aligned}\delta S_{\text{matter}} &= \int d^{2n}x \sqrt{G_{ab}} \delta G^{ab} T_{ab} &= \int d^4x \sqrt{G_{(4)}} \delta G_{(4)}^{\mu\nu} T_{\mu\nu}^{(4)} \\ &= \int d^{2n}x \sqrt{G_{ab}} (e^{-\sigma} \delta \gamma^{ab} - G^{ab} \delta \sigma) T_{ab} &= \int d^4x \sqrt{G_{(4)}} (e^{-\sigma_4} \delta \bar{\gamma}^{\mu\nu} - G_{(4)}^{\mu\nu} \delta \sigma_4) T_{\mu\nu}^{(4)}.\end{aligned}$$

At first sight, there appears to be a mismatch for the trace contribution, since  $\delta \sigma = \frac{1}{2(n-1)} \gamma_{ab} \delta \gamma^{ab}$  while  $\delta \sigma_4 = \frac{1}{2} \bar{\gamma}_{\mu\nu} \delta \bar{\gamma}^{\mu\nu}$ . This is resolved by noting that

$$\int d^{2n}x \sqrt{G_{ab}} T_{ab} G^{ab} = \int d^4x \sqrt{G_{(4)}} (n-1) T_{\mu\nu}^{(4)} G_{(4)}^{\mu\nu} \quad (5.6)$$

provided  $\varphi$  and  $\delta \bar{\gamma}^{\mu\nu}$  are constant along  $\mathcal{K}$ ; this is easily checked e.g. for scalar fields,

$$T_{ab} = \partial_a \varphi \partial_b \varphi - \frac{1}{2} G_{ab} (G^{cd} \partial_a \varphi \partial_b \varphi), \quad T_{\mu\nu}^{(4)} = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} G_{ab}^{(4)} (G_{(4)}^{cd} \partial_a \varphi \partial_b \varphi)$$

while  $T = 0$  for gauge fields. The last form of (5.4) implies as in (2.38) that the corresponding 4-dimensional Einstein tensor can be written as

$$\mathcal{G}_{(4)}^{\mu\nu} = \frac{1}{2} e^{-\sigma_4} \bar{\square}_{(4)} \delta \gamma^{\mu\nu} + \mathcal{O}(F), \quad (5.7)$$

using again a suitably adapted flat background, and using the harmonic gauge condition (5.10). For constant curvature compactifications as above, this reduces to

$$\begin{aligned}\mathcal{G}_{(4)}^{\mu\nu} &= e^{\sigma-2\sigma_4} \sum_i \frac{1}{r_i} \int_K \frac{d\zeta}{V_0} \bar{\square}(\gamma_{(i)}^{\mu\nu} \delta r_i) + \mathcal{O}(F) \\ &= e^{\sigma-2\sigma_4} \sum_i \frac{1}{r_i} \int_K \frac{d\zeta}{V_0} \gamma_{(i)}^{\mu\nu} \bar{\square} \delta r_i + \mathcal{O}(F) \\ &= \mathcal{P}_{(4)}^{\mu\nu; \rho\eta} T_{\rho\eta} + \mathcal{O}(F)\end{aligned} \quad (5.8)$$

assuming  $\nabla \gamma_{(i)} = 0$  or  $\partial|_p \delta r_i = 0$ , and<sup>25</sup> the  $2n$ -dimensional equation of motion (4.11) for the moduli. The  $\mathcal{O}(F)$  term will be discussed in the next section. Here

$$\mathcal{P}_{(4)}^{\mu\nu; \rho\eta} = \Lambda_0^{-4} e^{-2\sigma_4} \sum_i \frac{1}{r_i^2} \int_K \frac{d\zeta}{V_0} \gamma_{(i)}^{\mu\nu} \left( \gamma_{(i)}^{\rho\eta} - \frac{\dim \mathcal{K}_i}{2(n-1)} \gamma^{\rho\eta} \right). \quad (5.9)$$

In particular, we recover (4.16). The important point is that  $\mathcal{P}$  is now averaged over  $\mathcal{K}$ , and the partial metrics  $\gamma_{(i)}^{\mu\nu}$  probe different components of the energy-momentum tensor  $T_{\mu\nu}$ . This or a similar averaging might allow to make the present mechanism of brane gravity realistic.

Notice that we used above the 4-dimensional harmonic gauge condition

$$\partial^\nu h_{\mu\nu}^{(4)} - \frac{1}{2} \partial_\mu h^{(4)} = 0. \quad (5.10)$$

This follows again from the 4-dimensional equations of motion. Indeed, the variation of the action for any scalar field  $\varphi$  which is constant along  $\mathcal{K}$  can be written as

$$\delta_{(4)} S[\phi] = 2 \int d^4x \delta_{(4)} \varphi \partial_\nu (\sqrt{|G_{(4)}|} G_{(4)}^{\nu\mu} \partial_\mu \varphi) = 2 \int d^4x \sqrt{|G_{(4)}|} \delta_{(4)} \varphi \square_{(4)} \varphi \quad (5.11)$$

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<sup>25</sup>These manipulations are somewhat sketchy, and should be refined elsewhere.

Comparing with (5.1), it follows that the Laplace operator for the lowest KK modes reduces to that of  $G_{(4)}$ :

$$\sqrt{|G_{(4)}|} \square_G^{(4)} \varphi = V_0 \sqrt{|G_{(2n)}|} \square_G^{(2n)} \varphi. \quad (5.12)$$

In particular, the matrix coordinates  $x^\mu$  are harmonic also w.r.t.  $G_{(4)}$  in vacuum,

$$\Gamma_{(4)}^\mu = \square^{(4)} x^\mu = V_0 \frac{|G_{(2n)}|^{1/2}}{|G_{(4)}|^{1/2}} \square_G x^\mu = e^{\sigma-\sigma_4} \Gamma_{(2n)}^\mu = 0. \quad (5.13)$$

Therefore the harmonic gauge condition (5.10) holds.

We conclude that the massless moduli lead to (nearly) Ricci-flat deformations of the effective 4-dimensional metric, not only for toroidal compactifications but also under somewhat weaker assumptions on the compactification. The coupling of gravity to matter will quite generically lead to Newtonian gravity, and a (near-) Lorentz invariant 4-dimensional effective  $\mathcal{P}$  might be recovered upon averaging over  $\mathcal{K}$ , possibly leading to a viable gravity close to GR. A similar averaging may arise in the presence of multiple branes with intersecting compactifications  $\mathcal{K}$ , where each brane will contribute gravitational modes, which may couple to different components of the energy-momentum tensor. Such scenarios are very appealing also from the particle physics point of view, and are naturally realized in matrix models [15]. On the other hand, compactifications with  $\nabla\theta^{ab} \neq 0$  may also be interesting, as discussed next.

## 5.2 Non-constant $\theta^{ab}$

The toroidal compactification considered above are special because the Poisson tensor is covariantly constant. This implies that the metric fluctuations due to  $F$  couple only to derivatives of the e-m tensor. We briefly discuss the effect of more general compactifications with  $\nabla\theta^{ab} \neq 0$  on  $F$ . The equations of motion (3.6) now imply

$$\bar{\square} F_{ab} = 2\Lambda_0^{-4} e^{-\sigma} \left( T_{ef} \bar{G}^{fc} (\bar{\nabla}_b \bar{\nabla}_c \bar{\theta}^{ed} \bar{G}_{da} - \bar{\nabla}_a \bar{\nabla}_c \bar{\theta}^{ed} \bar{G}_{db}) + \mathcal{O}(\nabla T) \right). \quad (5.14)$$

Unlike for toroidal compactifications, the tangential fluctuations now couple to the e-m tensor and not just its derivatives, and may play a similar role as  $\phi^A$  for gravity. Computing the 4-dimensional Einstein tensor for the averaged metric

$$\begin{aligned} \mathcal{G}^{ab} = e^{-\sigma} \sum_i \frac{1}{r_i} \bar{\gamma}_{(i)}^{ab} \bar{\square} \delta r_i - \frac{1}{2} \bar{G}^{bd} \bar{\square} (\bar{\theta}^{ac} F_{cd}) - \frac{1}{2} \bar{G}^{ad} \bar{\square} (\bar{\theta}^{bc} F_{cd}) + \frac{1}{4} \bar{G}^{ab} \bar{\square} (\bar{\theta}^{cd} F_{cd}) \\ + \mathcal{O}(\nabla \nabla T) + \mathcal{O}(\delta^2) \end{aligned} \quad (5.15)$$

will lead to an Einstein-type equation

$$\mathcal{G}^{\mu\nu} = (\mathcal{P}_\phi^{\mu\nu;\rho\eta} + \mathcal{P}_F^{\mu\nu;\rho\eta}) T_{\rho\eta} + \mathcal{O}(\nabla \nabla T) + \mathcal{O}(\delta^2). \quad (5.16)$$

We assume that the  $\bar{\nabla} \bar{\theta} \bar{\nabla} F$  terms vanish upon averaging over  $\mathcal{K}$ , which needs to be verified in the specific compactifications. Here  $\mathcal{P}_\phi^{\mu\nu;\rho\eta}$  is as before, and

$$\mathcal{P}_F^{\mu\nu;\rho\eta} = \Lambda_0^{-4} e^{-\sigma} \mathcal{O}(\bar{\theta} \nabla \nabla \bar{\theta}) = \mathcal{O}(G_N) \quad (5.17)$$

describes the contribution of the would-be  $U(1)$  gauge fields due to the first term in (5.14). Then both transversal and tangential perturbations contribute with the same coupling strength  $G_N$ . Hence for such compactifications  $\mathcal{K}$  the tangential modes play a similar role as the transversal modes, and a 4-dimensional  $\mathcal{P}$  providing an appropriate coupling to the full 4-dimensional e-m tensor might be recovered upon averaging over  $\mathcal{K}$ . Thus after the dust has settled, the present scenario of emergent gravity from the IKKT model might provide a viable description of gravity and its quantization.

## 6 Conclusion

In this paper, a new mechanism is exhibited which leads to an effective 4-dimensional gravity on compactified brane solutions  $\mathcal{M}^4 \times \mathcal{K} \subset \mathbb{R}^{9,1}$  of the IKKT matrix model. Gravitational modes are encoded in the moduli of the compactification, which are transmitted to the non-compact space via the Poisson tensor. The required type of Poisson structure (dubbed split noncommutativity) arises naturally in the relevant solutions, so that the mechanism is very robust. No Einstein-Hilbert action is required, only the basic matrix model equations of motion are used which select a certain type of harmonic embedding. The Einstein tensor is indeed sourced by the energy-momentum tensor, however deviations from Ricci-flatness in vacuum are possible, and the gravitational coupling depends on the specific compactification  $\mathcal{K}$ . For the simplest case of  $\mathbb{R}^{3,1} \times T^2$ , the coupling is anisotropic. We argue that more sophisticated compactifications and/or a more complete treatment of the Poisson structure should allow to recover a physically viable effective gravity theory in 4 dimensions. The mechanism does not arise in a purely commutative setting although the model is very similar to  $\mathcal{N} = 4$  SYM, because the Poisson tensor  $\theta^{i\mu}$  plays an essential role to link the moduli of  $\mathcal{K}$  with the non-compact metric.

There are several reasons why this non-standard origin for gravity is interesting. First of all, it promises to give a perturbatively finite quantum theory of gravity. The reason is that the compactified backgrounds under consideration behaves in the UV like 4-dimensional spaces, due to their intrinsic noncommutative nature. This should imply perturbative finiteness, because the model can be viewed alternatively as  $\mathcal{N} = 4$  NC gauge theory on a 4-dimensional NC space. Furthermore, the model offers reasonable hope to resolve the cosmological constant problem, and we argued that vacuum energy induced by quantum mechanical zero-point fluctuations should be consistent with flat 4-dimensional geometries. Finally, this approach avoids the landscape problem in string theory, because it does not require 6-dimensional compactifications but only 2- or 4-dimensional compactification. Of course all these claims are bold, and require careful scrutiny and better justification. Nevertheless they appear to be reasonable, and certainly justify to study this matrix-model approach in detail.

This paper clearly leaves many open questions and loose ends. The main point is to demonstrate the mechanism, and to provide hints for further explorations. There are many obvious directions for follow-up work, in particular more sophisticated compactifications, non-trivial embeddings of the 4-dimensional space, and a more complete treatment of the tangential modes. The same mechanism should also be studied from other points of view, such as the BFSS model, or from a more stringy perspective. It remains to be seen if a fully realistic quantum theory of gravity and other fundamental interactions will emerge from this approach.

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## Appendix A: Extrinsic curvature

Consider the objects

$$K_{ij}^A = \nabla_i \partial_j x^A, \quad K_{ij;kl} = K_{ij}^A K_{kl}^B \eta_{AB} \quad (\text{A.1})$$

on  $\mathcal{M} \subset \mathbb{R}^m$ . Viewing the Cartesian embedding functions  $x^A$  as scalar fields on  $\mathcal{M}$ , we can consider  $K_{ij}^A$  as a rank 2 tensor field for each  $A$ , and  $K_{ij;kl}$  as a tensor field. If  $\nabla \equiv \nabla_G = \nabla_g$ , then the Gauss-Codazzi theorem states that

$$K_{ij;kl} - K_{jk;il} = R_{ikjl}. \quad (\text{A.2})$$

Let us compute these objects for the sphere  $S^m \subset \mathbb{R}^{m+1}$ . Using the  $SO(m+1)$  symmetry, we can use normal embedding coordinates  $x^i$ ,  $i = 1, \dots, m$  for

$$x^A = \begin{pmatrix} x^i, & i = 1, \dots, m \\ x^{m+1} = \sqrt{r^2 - \sum (x^i)^2} \end{pmatrix} \quad (\text{A.3})$$

at the north pole  $p = (0, \dots, 0, r)$ . Then

$$\partial_i \partial_j x^k = 0, \quad k = 1, \dots, m, \quad \partial_i \partial_j x^{m+1} = - \left( \frac{(x^{m+1})^2 \delta_{ij} + x^i x^j}{(x^{m+1})^3} \right), \quad (\text{A.4})$$

so that

$$\nabla_i \partial_j x^A|_p = P_N \partial_i \partial_j x^A|_p = \begin{pmatrix} 0 \\ -\frac{\delta_{ij}}{R} \end{pmatrix} \quad (\text{A.5})$$

which means that

$$K_{ij}^A = -\frac{g_{ij}}{r^2} x^A, \quad K_{ij;kl} = \frac{1}{r^2} g_{ij} g_{kl}. \quad (\text{A.6})$$

In particular for a torus  $T^m = \times_a S_{(a)}^1 \subset \mathbb{R}^{2m}$ , we obtain

$$K_{ij;kl} = \sum_a \frac{1}{r_a^2} g_{ij}^{(a)} g_{kl}^{(a)} \quad (\text{A.7})$$

where  $r_a$  are the radii of the cycles of  $T^m$ . One can then verify via the Gauss-Codazzi theorem that the intrinsic geometry is flat.

## Appendix B: Matrix energy-momentum tensor

We recall that the translations  $\delta X^A = c^A \mathbb{1}$  are symmetries of the matrix model. Adapting a standard trick from field theory, one can derive a corresponding conservation law by considering the following non-constant infinitesimal transformation<sup>26</sup>

$$\delta X^A = X^B [X^A, \varepsilon_B] + [X^A, \varepsilon_B] X^B \quad (\text{B.1})$$

where  $\varepsilon_B$  is an arbitrary matrix. As elaborated in [2], this leads to

$$\delta S_{\text{YM}} = -\text{Tr} \varepsilon_B [X_A, \mathcal{T}^{AB}] = \text{Tr} [X_A, \varepsilon_B] \mathcal{T}^{AB} \quad (\text{B.2})$$

for an arbitrary matrix (!)  $\varepsilon_B$ , so that

$$[X_B, \mathcal{T}^{AB}] = 0 \quad (\text{B.3})$$

where  $\mathcal{T}^{AB}$  is the "matrix" energy-momentum tensor. Its bosonic contribution is given explicitly by

$$\mathcal{T}^{AB} = \frac{1}{2}([X^A, X^C][X^B, X_C] + (A \leftrightarrow B)) - \frac{1}{4}\eta^{AB}[X^C, X^D][X_C, X_D]. \quad (\text{B.4})$$

Its  $U(1)$ -valued component  $\mathcal{T}^{AB} = \mathcal{T}_{\text{geom}}^{AB} + \mathcal{T}_{\text{nonabel}}^{AB}$  consists of a geometrical term

$$\mathcal{T}_{\text{geom}}^{AB} = \partial_a X^A \partial_b X^B \theta^{aa'} \theta^{bb'} T_{ab}^{\text{geom}}, \quad T_{ab}^{\text{geom}} = -g_{ab} + \frac{1}{4}G_{ab}(G^{cd}g_{cd}) \quad (\text{B.5})$$

plus the energy-momentum tensor of the nonabelian gauge and scalar fields. It is easy to see from (B.4) and (2.5) that this can be rewritten as

$$\{X_B, \mathcal{T}_{\text{geom}}^{AB}\} = \{X^A, X_B\} \square X^B = e^\sigma \theta^{ac} \partial_a x^A g_{bc} \square x^b = e^{2\sigma} G^{ab} G^{de} \nabla_e^{(g)} \theta_{db}^{-1} \partial_a x^A \quad (\text{B.6})$$

which defines a vector field on  $\mathcal{M}$ , using the identity

$$\square_G x^b = -\Gamma^b = \frac{1}{\sqrt{G}} \nabla_a^{(g)} (\sqrt{G} G^{ab}) = e^{-\sigma} \theta^{ac} g_{dc} \nabla_a^{(g)} \theta^{bd} = -G^{d'a} \nabla_a^{(g)} \theta_{b'd'}^{-1} \theta^{bb'} \quad (\text{B.7})$$

in the last step. On the other hand, in any local coordinates we can write

$$\begin{aligned} \{X_B, \mathcal{T}^{AB}\} &= \theta^{cd} \partial_c X^B \partial_d \mathcal{T}^{AD} \eta_{BD} \\ &= \theta^{cd} \partial_c X^B \partial_d (\partial_a X^A \partial_b X^D \theta^{aa'} \theta^{bb'} T_{a'b'}) \eta_{BD} \\ &= \theta^{cd} \partial_d (\partial_a X^A \partial_c X^B \partial_b X^D \theta^{aa'} \theta^{bb'} T_{a'b'}) \eta_{BD} \\ &= \partial_d (g_{cb} \theta^{bb'} T_{a'b'} \theta^{aa'} \partial_a X^A) \\ &= \sqrt{\theta} \partial_d (\sqrt{\theta^{-1}} \theta^{cd} g_{cb} \theta^{bb'} T_{a'b'} \theta^{aa'} \partial_a X^A) \\ &= \frac{e^\sigma}{\sqrt{G}} \partial_d (\sqrt{G} G^{db'} T_{a'b'} \theta^{aa'} \partial_a X^A) \end{aligned} \quad (\text{B.8})$$

using the identity (3.3). In NEC or equivalently  $\nabla^{(g)}$ , the double derivative term  $\partial_d \partial_a X^A$  is in the normal bundle, so that we obtain the identity

$$\nabla_d^{(g)} (\sqrt{G} G^{db} T_{a'b}^{\text{geom}} \theta^{aa'}) = e^\sigma \sqrt{G} G^{ab} G^{da} \nabla_a^{(g)} \theta_{db}^{-1}. \quad (\text{B.9})$$

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<sup>26</sup>it is not hard to see that this preserves the measure of the matrix path integral.



Finally, to see the relation with the usual e-m tensor, note that the nonabelian components contribute via

$$[X^a, X^b] = \theta^{aa'} \theta^{bb'} (\theta_{a'b'}^{-1} + \mathcal{F}_{a'b'}) \quad (\text{B.10})$$

and we recover the standard form of the e-m tensor for nonabelian gauge fields

$$\mathcal{T}_{\text{nonabel}}^{ab} = e^\sigma \theta^{aa'} \theta^{bb'} (\mathcal{F}_{a'c} G^{cc'} \mathcal{F}_{c'b'} - \frac{1}{4} G_{a'b'} (\mathcal{F}\mathcal{F})) = e^\sigma \theta^{aa'} \theta^{bb'} T_{a'b'}^{\text{nonabel}}. \quad (\text{B.11})$$

## Appendix C: Harmonic correction due to matter.

In the presence of matter, the equations of motion are modified so that the matrix coordinates are no longer harmonic,  $\square x^a \neq 0$ . However, we will show that the deviation from harmonicity is small and negligible compared with the energy-momentum tensor source for the Ricci tensor, justifying the above derivation of the geometric equations of motion. To see this, we recall the general expression (2.35) for the linearized Ricci tensor in terms of  $h_{ab}$ , which can be written as

$$\begin{aligned} \delta R^{ab} &= \frac{1}{2} \nabla^a (\nabla_d h^{bd} - \frac{1}{2} \partial^b h) + \frac{1}{2} \nabla^b (\nabla_d h^{ad} - \frac{1}{2} \partial^a h) - \frac{1}{2} \square h^{ab} + \mathcal{O}(R) \\ &= -\frac{1}{2} \nabla^a \square x^b - \frac{1}{2} \nabla^b \square x^a - \frac{1}{2} \square h^{ab} + \mathcal{O}(R) \end{aligned} \quad (\text{C.1})$$

since

$$\Gamma^a = -\square x^a \sim \nabla_d h^{ad} - \frac{1}{2} \partial^a h \quad (\text{C.2})$$

(note that  $h^{ab} = -\delta G^{ab}$ ). Now note that the  $U(1)$  component of the conservation law (B.3) casts the tangential equations of motion in the presence of matter in the following useful form

$$e^\sigma \Gamma^d = -e^\sigma \square x^d = g^{bd} \theta_{bc}^{-1} [X_B, \mathcal{T}_{\text{nonabel}}^{cB}], \quad (\text{C.3})$$

and similarly for the fermionic matter contributions. Plugging this into (C.1), the terms  $\nabla^a \square x^b$  contribute second derivatives of the energy-momentum tensor resp. of  $\theta^{ab}$ , which is much smaller than the matter contributions to  $\square h^{ab}$  which led to (??), thus finally justifying its derivation in the presence of matter (at least if  $\theta^{ab}$  is constant). Notice that these derivative contributions are of the same magnitude as the contributions of the tangential perturbations  $F$  in (??), so that the derivative corrections to the Einstein equations in the presence of matter require a more careful investigation.

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